FINITE GENERATORS FOR COUNTABLE GROUP ACTIONS IN THE BOREL AND BAIRE CATEGORY SETTINGS

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Abstract. For a countable group G and a standard Borel G-space X, a countable Borel partition $\mathcal{P}$ of X is called a *generator* if $G\mathcal{P} := \{gA : g \in G, A \in \mathcal{P}\}$ generates the Borel σ-algebra of X. For $G = \mathbb{Z}$, the Kolmogorov-Sinai theorem implies that if X admits an invariant probability measure with infinite entropy, then there is no finite generator. It was asked by Benjamin Weiss in '87 whether removing this obstruction would guarantee the existence of a finite generator; more precisely, if X does not admit an invariant probability measure at all, is there a finite generator? We give a positive answer to this question for arbitrary G in case X has a σ-compact topological realization (e.g. when X is a σ-compact Polish G-space).

Assuming a positive answer to Weiss's question for arbitrary Borel $\mathbb{Z}$-spaces, we prove two dichotomies, one of which states the following: for an aperiodic Borel $\mathbb{Z}$-space X, either X admits an invariant probability measure of infinite entropy or there is a finite generator.

We also show that finite generators always exist in the context of Baire category, thus answering positively a question raised by Kechris in the mid-'90s. More precisely, we prove that any aperiodic Polish G-space admits a 4-generator on an invariant comeager set.

It is not hard to prove that finite generators exist in the presence of a weakly wandering or even just a locally weakly wandering complete section. However, we develop a sufficient condition for the nonexistence of non-meager weakly wandering subsets of a Polish $\mathbb{Z}$-space, using which we show that the nonexistence of an invariant probability measure does not guarantee the existence of a countably generated Baire measurable partition into invariant sets that admit weakly wandering complete sections. This answers negatively a question posed in [EHN93], which was also independently answered by Ben Miller.

1. Introduction

§1. *Throughout the paper let G denote a countably infinite discrete group.* Let X be a Borel G-space, i.e. a standard Borel space equipped with a Borel action of G. Consider the following game: Player I chooses a finite or countable Borel partition $\mathcal{I} = \{A_n\}_{n<k}$ of X, $k \leq \infty$, then Player II chooses $x \in X$ and Player I tries to guess x by asking questions to Player II regarding which piece of the partition x lands in when moved by a certain group element. More precisely, for every $g \in G$, Player I asks to which A_n does gx belong and Player II gives $n_g < k$ as an answer. Whether or not Player I can uniquely determine x from the sequence $\{n_g\}_{g\in G}$ of responses depends on how cleverly he chose the partition $\mathcal{I}$. A partition is called a generator if it guarantees that Player I will determine x correctly no matter which x Player II chooses. Here is the precise definition, which also explains the terminology.

Definition 1.1 (Generator). *Let $k \leq \infty$ and $\mathcal{I} = \{A_n\}_{n<k}$ be a Borel partition of X (i.e. each A_n is Borel). $\mathcal{I}$ is called a generator if $G\mathcal{I} := \{gA_n : g \in G, n < k\}$ generates the Borel σ-algebra of X. We also call $\mathcal{I}$ a k-generator, and, if k is finite, a finite generator.*

For each $k \leq \infty$, we give k^G the product topology and let G act by shift on k^G. For a Borel partition $\mathcal{I} = \{A_n\}_{n<k}$ of X, let $f_{\mathcal{I}} : X \to k^G$ be defined by $x \mapsto (n_g)_{g \in G}$, where n_g is such that $gx \in A_{n_g}$. This is often called the symbolic representation map for the process $(X, G, \mathcal{I})$. Clearly $f_{\mathcal{I}}$ is a Borel G-map and, for every $x \in X$, $f_{\mathcal{I}}(x)$ is the sequence of responses of Player I in the above game. Based on this we have the following.

Observation 1.2. *Let $k \leq \infty$ and $\mathcal{I} = \{A_n\}_{n<k}$ be a Borel partition of X. The following are equivalent:*

(1) $\mathcal{I}$ is a generator.
(2) $G\mathcal{I}$ separates points, i.e. for all distinct $x, y \in X$ there is $A \in G\mathcal{I}$ such that $x \in A \not\Leftrightarrow y \in A$.
(3) $f_{\mathcal{I}}$ is one-to-one.

In all of the arguments below, we use these equivalent descriptions of a finite generator without comment.

Given a Borel G-map $f : X \to k^G$ for some $k \leq \infty$, define a partition $\mathcal{I}_f = \{A_n\}_{n<k}$ by $A_n = f^{-1}(V_n)$, where $V_n = \{\alpha \in k^G : \alpha(1_G) = n\}$. Note that $f_{\mathcal{I}_f} = f$. This and the above observation imply the following.

Observation 1.3. *For $k \leq \infty$, X admits a k-generator if and only if there is a Borel G-embedding of X into k^G.*

§2. Generators arose in the study of entropy in ergodic theory. Let (X, μ, T) be a dynamical system, i.e. (X, μ) is a standard probability space and T is a Borel measure preserving automorphism of X. We can interpret the above game as follows: X is the set of possible pictures of the world, $\mathcal{I}$ is an experiment that we are conducting and T is the unit of time. Assume that $\mathcal{I}$ is finite (indeed, we want our experiment to have finitely many possible outcomes). Player I repeats the experiment every day and Player II tells its outcome. The goal is to find the true picture of the world (i.e. $x \in X$ that Player II has in mind) with probability 1.

The entropy of the experiment $\mathcal{I}$ is defined by

$$h_\mu(\mathcal{I}) = -\sum_{n<k} \mu(A_n) \log \mu(A_n),$$

and intuitively, it measures our probabilistic uncertainty about the outcome of the experiment. For example, if for some $n < k$, A_n had probability 1, then we would be probabilistically certain that the outcome is going to be in A_n. Conversely, if all of A_n had probability $\frac{1}{k}$, then our uncertainty would be the highest. Equivalently, according to Shannon's interpretation, $h_\mu(\mathcal{I})$ measures how much information we gain from learning the outcome of the experiment.

We now define the time average of the entropy of $\mathcal{I}$ by

$$h_\mu(\mathcal{I}, T) = \lim_{n \to \infty} \frac{1}{n} h_\mu(\bigvee_{i<n} T^i \mathcal{I}),$$

where $\bigvee$ denotes the joint of the partitions (the least common refinement). The sequence in the limit is decreasing and hence the limit always exists and is finite (see [Gla03] or [Rud90]).

Finally the entropy of the dynamical system (X, μ, T) is defined as the supremum over all (finite) experiments:

$$h_\mu(T) = \sup_{\mathcal{I}} h_\mu(\mathcal{I}, T),$$

and it could be finite or infinite. Now it is plausible that if $\mathcal{I}$ is a finite generator (and hence Player I wins the above game), then $h_\mu(\mathcal{I}, T)$ should be all the information there is to obtain about X and hence $\mathcal{I}$ achieves the supremum above. This is indeed the case as the following theorem (Theorem 14.33 in [Gla03]) shows.

Theorem 1.4 (Kolmogorov-Sinai, '58-59)**.** *If $\mathcal{I}$ is a finite generator modulo μ-NULL, then $h_\mu(T) = h_\mu(\mathcal{I}, T)$. In particular, $h_\mu(T) \leq \log(|\mathcal{I}|) < \infty$*

Here μ-NULL denotes the σ-ideal of μ-null sets and, by definition, a statement holds modulo a σ-ideal $\mathfrak{I}$ if it holds on $X \setminus Z$, for some $Z \in \mathfrak{I}$. We will also use this for MEAGER, the σ-ideal of meager sets in a Polish space.

In case of ergodic systems, i.e. dynamical systems where every (measurable) invariant set is either null or co-null, the converse of Kolmogorov-Sinai theorem is true (see [Kri70]):

Theorem 1.5 (Krieger, '70)**.** *Suppose (X, μ, T) is ergodic. If $h_\mu(T) < \log k$, for some $k \geq 2$, then there is a k-generator modulo μ-NULL.*

§3. Now let X be just a Borel $\mathbb{Z}$-space with no measure specified. Then by the Kolmogorov-Sinai theorem, if there exists an invariant Borel probability measure on X with infinite entropy, then X does not admit a finite generator. What happens if we remove this obstruction? The following question was first stated in [Wei87] and restated in [JKL02] for an arbitrary countable group.

Question 1.6 (Weiss, '87)**.** *Let G be a countable group and X be a Borel G-space. If X does not admit any invariant Borel probability measure, does it have a finite generator?*

Assuming that the answer to this question is positive for $G = \mathbb{Z}$, we prove the following dichotomy:

Theorem 7.5. *Suppose the answer to Question 1.6 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant Borel probability measure with infinite entropy;
(2) X admits a finite generator.

We remark that the nonexistence of an invariant ergodic probability measure of infinite entropy does not guarantee the existence of a finite generator. For example, let X be a direct sum of uniquely ergodic actions $\mathbb{Z} \curvearrowright X_n$ such that the entropy h_n of each X_n is finite but $h_n \to \infty$. Then X does not admit an invariant ergodic probability measure with infinite entropy since otherwise it would have to be supported on one of the X_n, contradicting unique ergodicity. Neither does X admit a finite generator since that would contradict Krieger's theorem applied to X_n, for large enough n.

However, assuming again that the answer to 1.6 is positive for $G = \mathbb{Z}$, we prove the following dichotomy suggested by Kechris:

Theorem 7.3. *Suppose the answer to Question 1.6 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant ergodic Borel probability measure with infinite entropy,

(2) there exists a partition $\{Y_n\}_{n\in\mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator.

The proofs of these dichotomies presented in Section 7 use the Ergodic Decomposition Theorem and a version of Krieger's theorem together with Theorem 6.10 about separating the equivalence classes of a smooth equivalence relation.

Definition 1.7. *Let X be a Borel G-space and denote its Borel σ-algebra by $\mathfrak{B}(X)$. For a topological property P (e.g. Polish, σ-compact, etc.), we say that X admits a P topological realization, if there exists a Hausdorff second countable topology on X satisfying P such that it makes the G-action continuous and its induced Borel σ-algebra is equal to $\mathfrak{B}(X)$.*

We remark that every Borel G-space admits a Polish topological realization (this is actually true for an arbitrary Polish group, but it is a highly non-trivial result of Becker and Kechris, see 5.2 in [BK96]). The main result of this paper is a positive answer to Question 1.6 in case X has a σ-compact realization:

Theorem 4.5. *Let X be a Borel G-space that admits a σ-compact realization. If there is no G-invariant Borel probability measure on X, then X admits a Borel 32-generator.*

For example, 1.6 has a positive answer when G acts continuously on a locally compact or even σ-compact Polish space.

Before explaining the idea of the proof of the above theorem, we present previously known results as well as other related results obtained in this paper.

§4. In [Wei87] it was shown that every aperiodic (i.e. having no finite orbits) $\mathbb{Z}$-space admits a countable generator. This has been generalized to any countable group in [JKL02].

Theorem 1.8 (Jackson-Kechris-Louveau, '02)**.** *Every aperiodic Borel G-space X admits a countable generator. In particular, there is a Borel G-embedding of X into $\mathbb{N}^G$.*

Recall that this is sharp in the sense that we could not hope to obtain a finite generator solely from the aperiodicity assumption. Indeed, the Kolmogorov-Sinai theorem implies that dynamical systems with infinite entropy cannot have a finite generator, and there do exist continuous aperiodic actions of $\mathbb{Z}$ with infinite entropy (e.g. the action of $\mathbb{Z}$ on $[0,1]^{\mathbb{Z}} \setminus A$ by shift, where A is the set of periodic points and the measure is the product of the Lebesgue measure).

§5. The following result gives a positive answer to a version of Question 1.6 in the measure-theoretic context (see [Kre70] for $G = \mathbb{Z}$ and [Kun74] for arbitrary G).

Theorem 1.9 (Krengel, Kuntz, '74)**.** *Let X be a Borel G-space and let μ be a quasi-invariant Borel probability measure on X (i.e. G preserves the μ-null sets). If there is no invariant Borel probability measure absolutely continuous with respect to μ, then X admits a 2-generator modulo μ-NULL.*

The proof uses a version of the Hajian-Kakutani-Itô theorem (see [HK64] and [HI69]), which states that the hypothesis of the Krengel-Kuntz theorem is equivalent to the existence of a weakly wandering set (see Definition 9.1) of positive measure. We show in Section 9 that having a weakly wandering (or even just *locally* weakly wandering) set of full saturation implies the existence of finite generators in the Borel context (Theorem 9.5).

However, it was shown by Eigen-Hajian-Nadkarni in [EHN93] that the analogue of the Hajian-Kakutani-Itô theorem fails in the Borel context. In Section 10, we strengthen this result by showing that it fails even in the context of Baire category (Corollary 10.11). This result is a consequence of a criterion for non-existence of non-meager weakly wandering sets (Theorem 10.7), and it implies a negative answer to the following question asked in [EHN93] (question (ii) on page 9):

Question 1.10 (Eigen-Hajian-Nadkarni, '93)**.** *Let X be a Borel $\mathbb{Z}$-space. If X does not admit an invariant probability measure, is there a countably generated (by Borel sets) partition of X into invariant sets, each of which admits a weakly wandering set of full saturation?*

Ben Miller pointed out to us that a negative answer to this question could also be inferred from Propositions 3.6 and 3.7 of his PhD thesis (see [Mil08]). However, the implication is indirect and these propositions do not provide a criterion for non-existence of weakly wandering sets.

§6. In the mid-'90s, Kechris asked whether an analogue of the Krengel-Kuntz theorem holds in the context of Baire category (see 6.6.(B) in [JKL02]), more precisely:

Question 1.11 (Kechris, mid-'90s)**.** *If a countable group G acts continuously on a perfect Polish space X and the action is generically ergodic (i.e. every invariant Borel set is meager or comeager), does it follow that there is a finite generator on an invariant comeager set?*

Note that a positive answer to Question 1.6 for an arbitrary group G would imply a positive answer to this question because of the following theorem (cf. Theorem 13.1 in [KM04]):

Theorem 1.12 (Kechris-Miller, '04)**.** *Let X be a Polish G-space. If the action is aperiodic, then there is an invariant dense G_δ set $X' \subseteq X$ that does not admit an invariant Borel probability measure.*

We give an affirmative answer to Question 1.11 in Section 8; in fact, we prove the following slightly stronger result:

Theorem 8.2. *Let X be a Polish G-space. If X is aperiodic, then the action admits a 4-generator on an invariant comeager set.*

The proof of this uses the Kuratowski-Ulam method introduced in the proofs of Theorems 12.1 and 13.1 in [KM04]. This method was inspired by product forcing and its idea is as follows. Suppose we want to prove the existence of an object that satisfies a certain condition on a comeager set (in our case a finite partition). We give a parametrized construction of such objects A_α, where the parameter α ranges over $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ (or any other Polish space), and then try to show that for comeager many values of α, A_α has the desired property Φ on a comeager set. In other words, we want to prove $\forall^* \alpha \forall^* x \Phi(\alpha, x)$, where $\forall^*$ means "for comeager many". Now the key point is that the Kuratowski-Ulam theorem allows us to switch the order of the quantifiers and prove $\forall^* x \forall^* \alpha \Phi(\alpha, x)$ instead. The latter is often an easier task since it allows one to work locally with fixed x.

§7. We now briefly outline the idea of the proof of the main result (Theorem 4.5). First we present an equivalent condition to the nonexistence of invariant measures that is proved by Nadkarni in [Nad91] and is the analogue of Tarski's theorem about paradoxical decompositions (see [Wag93]) for countably additive measures.

Let X be a Borel G-space and denote the set of invariant Borel probability measures on X by $\mathcal{M}_G(X)$. Also, for $S \subseteq X$, let $[S]_G$ denote the saturation of S, i.e. $[S]_G = \bigcup_{g \in G} gS$.

The following definition makes no reference to any invariant measure on X, yet provides a sufficient condition for the measure of two sets to be equal (resp. $\leq$ or $<$).

Definition 1.13. *Two Borel sets $A, B \subseteq X$ are said to be equidecomposable (denoted by $A \sim B$) if there are Borel partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of A and B, respectively, and $\{g_n\}_{n \in \mathbb{N}} \subseteq G$ such that $g_n A_n = B_n$. We write $A \preceq B$ if $A \sim B' \subseteq B$, and we write $A \prec B$ if moreover $[B \setminus B']_G = [B]_G$.*

The following explains the above definition.

Observation 1.14. *Let $A, B \subseteq X$ be Borel sets.*

(a) If $A \sim B$, then $\mu(A) = \mu(B)$ for any $\mu \in \mathcal{M}_G(X)$.
(b) If $A \preceq B$, then $\mu(A) \leq \mu(B)$ for any $\mu \in \mathcal{M}_G(X)$.
(c) If $A \prec B$, then either $\mu(A) = \mu(B) = 0$ or $\mu(A) < \mu(B)$ for any $\mu \in \mathcal{M}_G(X)$.

Definition 1.15. *A Borel set $A \subseteq X$ is called compressible if $A \prec A$.*

It is clear from the observation above that if a Borel set $A \subseteq X$ is compressible, then $\mu(A) = 0$ for all $\mu \in \mathcal{M}_G(X)$. In particular, if X itself is compressible then $\mathcal{M}_G(X) = \emptyset$. Thus compressibility is an apparent obstruction to having an invariant probability measure. It turns out that it is the only one:

Theorem 1.16 (Nadkarni, '91)**.** *Let X be a Borel G-space. There is an invariant Borel probability measure on X if and only if X is not compressible.*

The proof of this first appeared in [Nad91] for $G = \mathbb{Z}$ and is also presented in Chapter 4 of [BK96] for an arbitrary countable group G. This theorem is what makes it "possible" to work with the hypothesis of Question 1.6 since it equates the nonexistence of a certain kind of object (an invariant probability measure) with the existence of another (the sets witnessing the compressibility of X).

For a finite Borel partition $\mathcal{I}$ of X, we define the notion of $\mathcal{I}$-compressibility, which basically means the compressibility of the image set under $f_{\mathcal{I}}$ (recall that $f_{\mathcal{I}}$ is the symbolic representation of X with respect to $\mathcal{I}$). Furthermore, for $i \in \mathbb{N}$, we define the notion of i-compressibility (Definition 2.12) and show that if X is i-compressible, then it admits a 2^{i+1}-generator (Proposition 2.30). In fact, under the assumption that X is compressible, having a finite generator turns out to be equivalent to X being i-compressible for some $i \geq 1$ (Corollary 2.35). This shows that i-compressibility is the right notion to look at when studying Question 1.6.

Now ideally we would like to prove that the notions of compressibility and i-compressibility coincide, at least for sufficiently large i, since this would imply a positive answer to Question 1.6. This turns out to be true when X has a σ-compact realization and we show this as follows. In the proof of Nadkarni's theorem, one assumes that X is not compressible and constructs an invariant probability measure. We give a similar construction for i-compressibility instead of compressibility, but unfortunately the proof yields only a *finitely additive* invariant probability measure (Corollary 3.16). However, with the additional assumption that X is σ-compact, we are able to concoct a countably additive invariant measure out of it (Corollary 4.4), and thus obtain Theorem 4.5.

§8. Lastly, we give a positive answer to a version of Question 1.6 with slightly stronger hypothesis. It is not hard to prove (see 2.24) that for a Borel G-space X, the nonexistence of invariant probability measures on X is equivalent to the existence of so-called traveling sets of full saturation (Definition 2.18). We define a slightly stronger notion of a locally finitely traveling set (Definition 5.2), and show in 5.5 that if there exists such a set of full saturation, then X admits a 32-generator. The proof uses the machinery discussed in §7.

§9 Organization of the paper. In Section 2, we develop the theory of i-compressibility and establish its connection with the existence of finite generators. More particularly, in Subsection 2.1 we give the definition of $\mathcal{I}$-equidecomposability and prove the important property of orbit-disjoint countable additivity (see 2.9), which is what makes $\mathfrak{C}_i$ (defined below) a σ-ideal. In Subsections 2.2 and 2.3 we define the notions of i-compressibility and i-traveling sets and establish their connection. Finally, in Subsection 2.4, we show how to construct a finite generator using an i-traveling complete section (by definition, a complete section is a set that meets every orbit, equivalently, has full saturation).

In Section 3, we prove the main theorem, which provides means of constructing finitely additive invariant measures that are non-zero on a given non-i-compressible set. In the following two sections we establish two corollaries of this theorem, namely 4.5 and 5.5, where the former is the main result of the paper stated above and the latter is the result discussed in §8.

In Section 6, we show that given a smooth equivalence relation E on X with $E \supseteq E_G$, there exists a finite partition $\mathcal{P}$ such that $G\mathcal{P}$ separates points in different classes of E; in fact, we give an explicit construction of such $\mathcal{P}$. This result is then used in the following section, where we establish the potential dichotomy theorems mentioned above (7.3 and 7.5).

Section 8 establishes the existence of a 4-generator on an invariant comeager set, and Section 9 provides various examples of i-compressible actions involving locally weakly wandering sets. Finally, in Section 10 we develop a criterion for non-existence of non-meager weakly wandering sets and derive a negative answer to Question 1.10.

§10 Open questions. Here are some open questions that arose in this research. Let X denote a Borel G-space.

(A) Is X being compressible equivalent to X being i-compressible for some $i \geq 1$?
(B) Does the existence of a traveling complete section imply the existence of a locally finitely traveling complete section?

A positive answer to any of these questions would imply a positive answer to Question 1.6 since (A) is just a rephrasing of Question 1.6 because of 2.35 and for (B), it follows from 2.22 and 5.5.

In the original version of the paper, it was also asked whether X always admits a σ-compact realization. However, this was answered negatively by Conley, Kechris and Miller.

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2. Finite generators and i-compressibility

Throughout this section let X be a Borel G-space and E_G be the orbit equivalence relation on X. For $A \subseteq X$ and G-invariant $P \subseteq X$, let $A^P := A \cap P$.

For an equivalence relation E on X and $A \subseteq X$, let $[A]_E$ denote the saturation of A with respect to E, i.e. $[A]_E = \{x \in X : \exists y \in A(xEy)\}$. In case $E = E_G$, we use $[A]_G$ instead of $[A]_{E_G}$.

Let $\mathfrak{B}$ denote the class of all Borel sets in standard Borel spaces and let Γ be a σ-algebra of subsets of standard Borel spaces containing $\mathfrak{B}$ and closed under Borel preimages. For example, $\Gamma = \mathfrak{B}$, $\sigma(\mathbf{\Sigma}_1^1)$, universally measurable sets. For $A \subseteq X$, let $\Gamma(A)$ denote the set of Γ sets relative to A, i.e. $\Gamma(A) = \{B \cap A : B \subseteq X, B \in \Gamma\}$.

2.1. The notion of $\mathcal{I}$-equidecomposability

A countable partition of X is called Borel if all the sets in it are Borel. For a finite Borel partition $\mathcal{I} = \{A_i : i < k\}$ of X, let $F_{\mathcal{I}}$ denote the equivalence relation of not being separated by $G\mathcal{I} := \{gA_i : g \in G, i < k\}$, more precisely, $\forall x, y \in X$,

$$xF_{\mathcal{I}}y \Leftrightarrow f_{\mathcal{I}}(x) = f_{\mathcal{I}}(y),$$

where $f_{\mathcal{I}}$ is the symbolic representation map for $(X, G, \mathcal{I})$ defined above. Note that if $\mathcal{I}$ is a generator, then $F_{\mathcal{I}}$ is just the equality relation.

For an equivalence relation E on X and $A, B \subseteq X$, A is said to be E-invariant relative to B or just $E\downharpoonright_B$-invariant if $[A]_E \cap B = A \cap B$.

Definition 2.1 ($\mathcal{I}$-equidecomposability)**.** *Let $A, B \subseteq X$, and $\mathcal{I}$ be a finite Borel partition of X. A and B are said to be equidecomposable with Γ pieces (denote by $A \sim^{\Gamma} B$) if there are $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ and partitions $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ of A and B, respectively, such that for all $n \in \mathbb{N}$*

- $g_nA_n = B_n$,
- $A_n \in \Gamma(A)$ *and* $B_n \in \Gamma(B)$.

If moreover,

- A_n *and* B_n *are* $F_{\mathcal{I}}$*-invariant relative to* A *and* B*, respectively,*

then we will say that A and B are $\mathcal{I}$-equidecomposable with Γ pieces and denote it by $A \sim_{\mathcal{I}}^{\Gamma} B$. If $\Gamma = \mathfrak{B}$, we will not mention Γ and will just write $\sim$ and $\sim_{\mathcal{I}}$.

Note that for any $\mathcal{I}$, A, B as above, A and B are $\mathcal{I}$-equidecomposable if and only if $f_{\mathcal{I}}(A)$ and $f_{\mathcal{I}}(B)$ are equidecomposable (although the images of Borel sets under $f_{\mathcal{I}}$ are analytic, they are Borel relative to $f_{\mathcal{I}}(X)$ due to the Lusin Separation Theorem for analytic sets). Also note that if $\mathcal{I}$ is a generator, then $\sim_{\mathcal{I}}$ coincides with $\sim$.

Observation 2.2. *Below let $\mathcal{I}, \mathcal{I}_0, \mathcal{I}_1$ denote finite Borel partitions of X, and $A, B, C \in \Gamma(X)$.*

(a) (Quasi-transitivity) If $A \sim_{\mathcal{I}_0}^{\Gamma} B \sim_{\mathcal{I}_1}^{\Gamma} C$, then $A \sim_{\mathcal{I}}^{\Gamma} C$ with $\mathcal{I} = \mathcal{I}_0 \vee \mathcal{I}_1$ (the least common refinement of $\mathcal{I}_0$ and $\mathcal{I}_1$).

(b) ($F_{\mathcal{I}}$-disjoint countable additivity) Let $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}}$ be partitions of A and B, respectively, into Γ sets such that $\forall n \neq m$, $[A_n]_{F_{\mathcal{I}}} \cap [A_m]_{F_{\mathcal{I}}} = [B_n]_{F_{\mathcal{I}}} \cap [B_m]_{F_{\mathcal{I}}} = \emptyset$. If $\forall n \in \mathbb{N}$, $A_n \sim_{\mathcal{I}}^{\Gamma} B_n$, then $A \sim_{I}^{\Gamma} B$.

If $A \sim B$, then there is a Borel isomorphism ϕ of A onto B with $\phi(x) E_G x$ for all $x \in A$; namely $\phi(x) = g_n x$ for all $x \in A_n$, where A_n, g_n are as in Definition 1.13. It is easy to see that the converse is also true, i.e. if such ϕ exists, then $A \sim B$. In Proposition 2.5 we prove the analogue of this for $\sim_{\mathcal{I}}^{\Gamma}$, but first we need the following lemma and definition that take care of definability and $F_{\mathcal{I}}$-invariance, respectively.

For a Polish space Y, $f : X \to Y$ is said to be Γ-measurable if the preimages of open sets under f are in Γ. For $A \in \Gamma(X)$ and $h : A \to G$, define $\hat{h} : A \to X$ by $x \mapsto h(x)x$.

Lemma 2.3. *If $h : A \to G$ is Γ-measurable, then the images and preimages of sets in Γ under $\hat{h}$ are in Γ.*

Proof. Let $B \subseteq A$, $C \subseteq X$ be in Γ. For $g \in G$, set $A_g = h^{-1}(g)$ and note that $\hat{h}(B) = \bigcup_{g \in G} g(A_g \cap B)$ and $\hat{h}^{-1}(C) = \bigcup_{g \in G} g^{-1}(gA_g \cap C)$. Thus $\hat{h}(B)$ and $\hat{h}^{-1}(C)$ are in Γ by the assumptions on Γ. □

The following technical definition is needed in the proofs of 2.5 and 2.9.

Definition 2.4. *For $A \subseteq X$ and a finite Borel partition $\mathcal{I}$ of X, we say that $\mathcal{I}$ is A-sensitive or that A respects $\mathcal{I}$ if A is $F_{\mathcal{I}}$-invariant relative to $[A]_G$, i.e. $[A]_{F_{\mathcal{I}}}^{[A]_G} = A$.*

For example, if $\mathcal{I}$ is finer than $\{A, A^c\}$, then $\mathcal{I}$ is A-sensitive. Note that if $A \sim_{\mathcal{I}} B$ and A respects $\mathcal{I}$, then so does B.

Proposition 2.5. *Let $A, B \in \Gamma(X)$ and let $\mathcal{I}$ be a Borel partition of X that is A-sensitive. Then, $A \sim_{\mathcal{I}}^{\Gamma} B$ if and only if there is an $F_{\mathcal{I}}$-invariant Γ-measurable map $\gamma : A \to G$ such that $\hat{\gamma}$ is a bijection between A and B. We refer to such γ as a witnessing map for $A \sim_{\mathcal{I}}^{\Gamma} B$. The same holds if we delete "$F_{\mathcal{I}}$-invariant" and "$\mathcal{I}$" from the statement.*

Proof. $\Rightarrow$: If $\{g_n\}_{n \in \mathbb{N}}$, $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ are as in Definition 2.1, then define $\gamma : A \to G$ by setting $\gamma|_{A_n} \equiv g_n$.
$\Leftarrow$: Let γ be as in the lemma. Fixing an enumeration $\{g_n\}_{n \in \mathbb{N}}$ of G with no repetitions, put $A_n = \gamma^{-1}(g_n)$ and $B_n = g_n A_n$. It is clear that $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ are partitions of A and B, respectively, into Γ sets. Since γ is $F_{\mathcal{I}}$-invariant, each A_n is $F_{\mathcal{I}}$-invariant relative to A and hence relative to $P := [A]_G = [B]_G$ because A respects $\mathcal{I}$. It remains to show that each B_n is $F_{\mathcal{I}}$-invariant relative to B. To this end, let $y \in [B_n]_{F_{\mathcal{I}}} \cap B$ and thus there is $x \in A_n$ such that $y F_{\mathcal{I}} g_n x$. Hence $z := g_n^{-1} y \ F_{\mathcal{I}} \ g_n^{-1} g_n x = x$ and therefore $z \in A_n$ because A_n is $F_{\mathcal{I}}$-invariant relative to P. Thus $y = g_n z \in B_n$. □

In the rest of the subsection we work with $\Gamma = \mathfrak{B}$.

Next we prove that $\mathcal{I}$-equidecomposability can be extended to $F_{\mathcal{I}}$-invariant Borel sets. First we need the following reflection principle.

Lemma 2.6 (A reflection principle)**.** *Let E be a Borel equivalence relation on X and $B \subseteq X$ be Borel. Define the predicate $\Phi \subseteq Pow(X)$ as follows:*

$$\Phi(A) \Leftrightarrow A \subseteq B \wedge A \textit{ is } E\textit{-invariant.}$$

If A is analytic and $\Phi(A)$ then there exists a Borel set $A' \supseteq A$ with $\Phi(A')$.

Proof. Define the predicate $\Psi \subseteq Pow(X)$ as follows:

$$\Psi(D) \Leftrightarrow B^c \subseteq D \wedge D \text{ is } E\text{-invariant.}$$

It is clear that $\Phi(D) \Leftrightarrow \Psi(D^c)$, so it is enough to show that if D is co-analytic and $\Psi(D)$, then there is a Borel set $D' \subseteq D$ with $\Psi(D')$.

Note that

$$\Psi(D) \Leftrightarrow \forall x \in X \forall y \in X (x \notin D \wedge y \in D \Rightarrow x \notin B \wedge \neg x E y).$$

Thus, setting $R(x,y) \Leftrightarrow x \notin B \wedge \neg x E y$, we apply The Burgess Reflection Theorem (see 35.18 in [Kec95]) to Ψ with $\Gamma = \mathbf{\Pi}^1_1$ and $A = D$, and get a Borel $D' \subseteq D$ with $\Psi(D')$. $\square$

Proposition 2.7 ($F_{\mathcal{I}}$-invariant extensions)**.** *If for some Borel partition $\mathcal{I}$ of X and Borel sets $A, B \subseteq X$, $A \sim_{\mathcal{I}} B$, then there exists Borel sets $A' \supseteq A$ and $B' \supseteq B$ such that A', B' are $F_{\mathcal{I}}$-invariant and $A' \sim_{\mathcal{I}} B'$. In fact, if $\{g_n\}_{n\in\mathbb{N}}, \{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}}$ witness $A \sim_{\mathcal{I}} B$, then there are $F_{\mathcal{I}}$-invariant Borel partitions $\{A'_n\}_{n\in\mathbb{N}}, \{B'_n\}_{n\in\mathbb{N}}$ of A' and B' respectively, such that $g_n A'_n = B'_n$ and $A'_n \supseteq A_n$ (and hence $B'_n \supseteq B_n$).*

Proof. Let $\{g_n\}_{n\in\mathbb{N}}, \{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}}$ be as in Definition 2.1 and put $\bar{A}_n = [A_n]_{F_{\mathcal{I}}}$. It is easy to see that for $n \neq m \in \mathbb{N}$,

(i) $\bar{A}_n \cap \bar{A}_m = \emptyset$;
(ii) $g_n \bar{A}_n \cap g_m \bar{A}_m = \emptyset$.

Put $\bar{A} = [A]_{F_{\mathcal{I}}}$ and note that $\{\bar{A}_n\}_{n\in\mathbb{N}}$ is a partition of $\bar{A}$. Although $\bar{A}_n$ and $\bar{A}$ are $F_{\mathcal{I}}$-invariant, they are analytic and in general not Borel. We obtain Borel analogues using $\mathbf{\Pi}^1_1$-reflection theorems.

Set $U = \bigcup_{n\in\mathbb{N}} (n \times \bar{A}_n)$ and define a predicate $\Phi \subseteq Pow(\mathbb{N} \times X)$ as follows:

$$\Phi(W) \Leftrightarrow \forall n (W_n \text{ is } F_{\mathcal{I}}\text{-invariant}) \wedge \forall n \neq m (W_n \cap W_m = \emptyset \wedge g_n W_n \cap g_m W_m = \emptyset),$$

where $W_n = \{x \in X : (n,x) \in W\}$, the section of W at n. Note that $\Phi(U)$.

Claim. *There is a Borel set $U' \supseteq U$ with $\Phi(U')$.*

Proof of Claim. For $W \subseteq \mathbb{N} \times X$, let

$$\Lambda(W) \Leftrightarrow \forall n \neq m (W_n \cap W_m = \emptyset \wedge g_n W_n \cap g_m W_m = \emptyset)$$

Note that

$$\Lambda(W) \Leftrightarrow \forall n \neq m \forall x \in X [(x \notin W_n \vee x \notin W_m) \wedge (x \notin g_n W_n \vee x \notin g_m W_m)].$$

Thus Λ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and hence, by the dual form of the First Reflection Theorem for $\mathbf{\Pi}^1_1$ (see the discussion following 35.10 in [Kec95]), there is a Borel set $V \supseteq U$ with $\Lambda(V)$, since $\Lambda(U)$.

Now applying Lemma 2.6 to $E = F_{\mathcal{I}}$, $B = V_n$ and $A = U_n$, we get $F_{\mathcal{I}}$-invariant Borel sets $U'_n \subseteq X$ with $U_n \subseteq U'_n \subseteq V_n$. Thus $U' := \bigcup_{n\in\mathbb{N}} (n \times U'_n)$ is what we wanted. $\dashv$

Put $A'_n = U'_n$ and $A' = \bigcup_{n\in\mathbb{N}} A'_n$. Thus $\{A'_n\}_{n\in\mathbb{N}}$ is a partition of A' into $F_{\mathcal{I}}$-invariant Borel sets. Also, $A_n \subseteq \bar{A}_n \subseteq A'_n$ and hence $A \subseteq \bar{A} \subseteq A'$. Put $B'_n = g_n A'_n$ and $B' = \bigcup_{n\in\mathbb{N}} B'_n$; thus $\{B'_n\}_{n\in\mathbb{N}}$ is a Borel partition of B'. Also note that B'_n are $F_{\mathcal{I}}$-invariant and $B' \supseteq B$ since A'_n are $F_{\mathcal{I}}$-invariant and $A_n \subseteq A'_n$. Thus $A' \sim_{\mathcal{I}} B'$ and we are done. $\square$

Lemma 2.8 (Orbit-disjoint unions)**.** *Let $A_k, B_k \in \mathfrak{B}(X)$, $k = 0, 1$, be such that $[A_0]_G$ and $[A_1]_G$ are disjoint and put $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$. If $\mathcal{I}$ is an A, B-sensitive finite Borel partition of X such that $A_k \sim_{\mathcal{I}} B_k$ for $k = 0, 1$, then $A \sim_{\mathcal{I}} B$. Moreover, if $\gamma_0 : A_0 \to G$ is a Borel map witnessing $A_0 \sim_{\mathcal{I}} B_0$, then there exists a Borel map $\gamma : A \to G$ extending γ_0 that witnesses $A \sim_{\mathcal{I}} B$.*

Proof. First assume without loss of generality that $X = [A]_G$ ($= [B]_G$) since the statement of the lemma is relative to $[A]_G$. Thus A, B are $F_{\mathcal{I}}$-invariant.

Applying 2.7 to $A_0 \sim_{\mathcal{I}} B_0$, we get $F_{\mathcal{I}}$-invariant $A_0' \supseteq A_0, B_0' \supseteq B_0$ such that $A' \sim_{\mathcal{I}} B'$. Moreover, by the second part of the same lemma, if $\gamma_0 : A_0 \to G$ is a witnessing map for $A_0 \sim_{\mathcal{I}} B_0$, then there is a witnessing map $\delta : A_0' \to G$ for $A' \sim_{\mathcal{I}} B'$ extending γ_0. Put $C = A_0' \cap A$ and note that C is $F_{\mathcal{I}}$-invariant since so are A_0' and A. Finally, put $\bar{A}_0 = \{x \in C : C^{[x]_G} = A^{[x]_G} \wedge \hat{\delta}(C^{[x]_G}) = B^{[x]_G}\}$ and note that $\bar{A}_0 \supseteq A_0$ since $\delta \supseteq \gamma_0$ and $[A_0]_G \cap [A_1]_G = \emptyset$.

Claim. *$\bar{A}_0$ is $F_{\mathcal{I}}$-invariant.*

Proof of Claim. First note that for any $F_{\mathcal{I}}$-invariant $D \subseteq X$ and $z \in X$, $[D^{[z]_G}]_{F_{\mathcal{I}}} = D^{[[z]_{F_{\mathcal{I}}}]_G}$. Furthermore, if $D \subseteq C$, then $[\hat{\delta}(D)]_{F_{\mathcal{I}}} = \hat{\delta}([D]_{F_{\mathcal{I}}})$ since $\hat{\delta}$ and its inverse map $F_{\mathcal{I}}$-invariant sets to $F_{\mathcal{I}}$-invariant sets.

Now take $x \in \bar{A}_0$ and let $Q = [[x]_{F_{\mathcal{I}}}]_G$. Since A, B, C are $F_{\mathcal{I}}$-invariant, $C^Q = [C^{[x]_G}]_{F_{\mathcal{I}}} = [A^{[x]_G}]_{F_{\mathcal{I}}} = A^Q$. Furthermore, $\hat{\delta}(C^Q) = \hat{\delta}([C^{[x]_G}]_{F_{\mathcal{I}}}) = [\hat{\delta}(C^{[x]_G})]_{F_{\mathcal{I}}} = [B^{[x]_G}]_{F_{\mathcal{I}}} = B^Q$. Thus, $\forall y \in [x]_{F_{\mathcal{I}}}$, $C^{[y]_G} = A^{[y]_G}$ and $\hat{\delta}(C^{[y]_G}) = B^{[y]_G}$; hence $[x]_{F_{\mathcal{I}}} \subseteq \bar{A}_0$. ⊣

Put $\bar{A}_1 = A \setminus \bar{A}_0$, $\alpha_0 = \delta\restriction_{\bar{A}_0}$, $\alpha_1 = \gamma_1\restriction_{\bar{A}_1}$, where γ_1 is a witnessing map for $A_1 \sim_{\mathcal{I}} B_1$. It is clear from the definition of $\bar{A}_0$ that $\bar{A}_0$ is E_G-invariant relative to A and hence $[\bar{A}_0]_G \cap [\bar{A}_1]_G = \emptyset$. Thus, for $k = 0, 1$, it follows that α_k witnesses $\bar{A}_k \sim_{\mathcal{I}} \bar{B}_k$, where $\bar{B}_k = \hat{\alpha_k}(\bar{A}_k)$. Furthermore, it is clear that $B^{[\bar{A}_k]_G} = \bar{B}_k$ and, since $[\bar{A}_0]_G \cup [\bar{A}_1]_G = X$, $\bar{B}_0 \cup \bar{B}_1 = B$. Now since $\bar{A}_k$ are $F_{\mathcal{I}}$-invariant, $\gamma = \alpha_0 \cup \alpha_1$ is $F_{\mathcal{I}}$-invariant and hence witnesses $A \sim_{\mathcal{I}} B$. Finally, $\alpha_0\restriction_{A_0} = \delta\restriction_{A_0} = \gamma_0$ and hence $\alpha_0 \supseteq \gamma_0$. □

Proposition 2.9 (Orbit-disjoint countable unions)**.** *For $k \in \mathbb{N}$, let $A_k, B_k \in \mathfrak{B}(X)$ be such that $[A_k]_G$ are disjoint and put $A = \bigcup_{k\in\mathbb{N}} A_k$, $B = \bigcup_{k\in\mathbb{N}} B_k$. Suppose that $\mathcal{I}$ is an A, B-sensitive finite Borel partition of X such that $A_k \sim_{\mathcal{I}} B_k$ for all k. Then $A \sim_{\mathcal{I}} B$.*

Proof. We recursively apply Lemma 2.8 as follows. Put $\bar{A}_n = \bigcup_{k\leq n} A_k$ and $\bar{B}_n = \bigcup_{k\leq n} B_k$. Inductively define Borel maps $\gamma_n : \bigcup_{k\leq n} A_k \to G$ such that γ_n is a witnessing map for $\bar{A}_n \sim_{\mathcal{I}} \bar{B}_n$ and $\gamma_n \sqsubseteq \gamma_{n+1}$. Let γ_0 be a witnessing map for $A_0 \sim_{\mathcal{I}} B_0$. Assume γ_n is defined. Then γ_{n+1} is provided by Lemma 2.8 applied to $\bar{A}_n$ and A_{n+1} with γ_n as a witness for $\bar{A}_n \sim_{\mathcal{I}} \bar{B}_n$. Thus $\gamma_n \sqsubseteq \gamma_{n+1}$ and γ_{n+1} witnesses $\bar{A}_{n+1} \sim_{\mathcal{I}} \bar{B}_{n+1}$.

Now it just remains to show that $\gamma := \bigcup_{n\in\mathbb{N}} \gamma_n$ is $F_{\mathcal{I}}$-invariant since then it follows that γ witnesses $A \sim_{\mathcal{I}} B$. Let $x, y \in A$ be $F_{\mathcal{I}}$-equivalent. Then there is n such that $x, y \in \bar{A}_n$. By induction on n, γ_n is $F_{\mathcal{I}}$-invariant and, since $\gamma\restriction_{\bar{A}_n} = \gamma_n$, $\gamma(x) = \gamma(y)$. □

Corollary 2.10 (Finite quasi-additivity)**.** *For $k = 0, 1$, let $A_k, B_k \in \mathfrak{B}(X)$ be such that $A_0 \cap A_1 = B_0 \cap B_1 = \emptyset$ and put $A = A_0 \cup A_1$, $B = B_0 \cup B_1$. Let $\mathcal{I}_k$ be an A_k, B_k-sensitive finite Borel partition of X. If $A_0 \sim_{\mathcal{I}_0} B_0$ and $A_1 \sim_{\mathcal{I}_1} B_1$, then $A \sim_{\mathcal{I}_0\vee\mathcal{I}_1} B$.*

Proof. Put $\mathcal{I} = \mathcal{I}_0 \vee \mathcal{I}_1$, $P = [A_0]_G \cap [A_1]_G$, $Q = [A_0]_G \setminus [A_1]_G$ and $R = [A_1]_G \setminus [A_0]_G$. Then A_k^P, B_k^P respect $\mathcal{I}$, and thus $[A_0]_{F_{\mathcal{I}}}^P \cap [A_1]_{F_{\mathcal{I}}}^P = \emptyset$, $[B_0]_{F_{\mathcal{I}}}^P \cap [B_1]_{F_{\mathcal{I}}}^P = \emptyset$. Hence $A^P \sim_{\mathcal{I}} B^P$ since the sets that are $F_{\mathcal{I}}$-invariant relative to A_k^P are also $F_{\mathcal{I}}$-invariant relative to A^P, and the same is true for B_k^P and B^P. Also, $A^Q \sim_{\mathcal{I}} B^Q$ and $A^R \sim_{\mathcal{I}} B^R$ because $A^Q = A_0$, $B^Q = B_0$, $A^R = A_1$, $B^R = B_1$. Now since P, Q, R are pairwise disjoint, it follows from Proposition 2.9 that $A \sim_{\mathcal{I}} B$. □

2.2. **The notion of i-compressibility**

For a finite collection $\mathcal{F}$ of subsets of X, let $<\mathcal{F}>$ denote the partition of X generated by $\mathcal{F}$.

Definition 2.11 (i-equidecomposibility)**.** *For $i \geq 1$, $A, B \subseteq X$, we say that A and B are i-equidecomposable with Γ pieces (write $A \sim_i^\Gamma B$) if there is an A-sensitive partition $\mathcal{I}$ of X generated by i Borel sets such that $A \sim_{\mathcal{I}}^\Gamma B$. For a collection $\mathcal{F}$ of Borel sets, we say that $\mathcal{F}$ witnesses $A \sim_i^\Gamma B$ if $|\mathcal{F}| = i$, $\mathcal{I} :=<\mathcal{F}>$ is A-sensitive and $A \sim_{\mathcal{I}}^\Gamma B$.*

Remark. In the above definition, it might seem more natural to have i be the cardinality of the partition $\mathcal{I}$ instead of the cardinality of the collection $\mathcal{F}$ generating $\mathcal{I}$. However, our definition above of i-equidecomposability is needed in order to show that the collection $\mathfrak{C}_i$ defined below forms a σ-ideal. More precisely, the presence of $\mathcal{F}$ is needed in the definition of i^*-compressibility, which ensures that the partition $\mathcal{I}$ in the proof of 2.17 is B-sensitive.

For $i \geq 1$, $A, B \subseteq X$, we write $A \preceq_i^\Gamma B$ if there is a Γ set $B' \subseteq B$ such that $A \sim_i^\Gamma B'$. If moreover $[A \setminus B]_G = [A]_G$, then we write $A \prec_i^\Gamma B$. If $\Gamma = \mathfrak{B}$, we simply write $\sim_i, \preceq_i, \prec_i$.

Definition 2.12 (i-compressibility)**.** *For $i \in \mathbb{N}$, $A \subseteq X$, we say that A is i-compressible with Γ pieces if $A \prec_i^\Gamma A$.*

Unless specified otherwise, we will be working with $\Gamma = \mathfrak{B}$, in which case we simply say i-compressible.

For a collection of sets $\mathcal{F}$ and a G-invariant set P, set $\mathcal{F}^P = \{A^P : A \in \mathcal{F}\}$. We will use the following observations without mentioning.

Observation 2.13. *Let $i, j \geq 2$, $A, A', B, B', C \in \mathfrak{B}$. Let $P \subseteq [A]_G$ denote a G-invariant Borel set and $\mathcal{F}, \mathcal{F}_0, \mathcal{F}_1$ denote finite collections of Borel sets.*

(a) If $A \sim_i B$ then $A^P \sim_i B^P$.
(b) If $\mathcal{F}$ witnesses $A \sim_i B$, then so does $\mathcal{F}^{[A]_G}$.
(c) If $A \sim_i B \sim_j C$, then $A \sim_{(i+j)} C$. In fact, $\mathcal{F}_0$ and $\mathcal{F}_1$ witness $A \sim_i B$ and $B \sim_j C$, respectively, then $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ witnesses $A \sim_{(i+j)} C$.
(d) If $A \preceq_i B \preceq_j C$, then $A \preceq_{(i+j)} C$. If one of the first two $\preceq$ is $\prec$ then $A \prec_{(i+j)} C$.
(e) If $A \sim_i B$ and $A' \sim_j B'$ with $A \cap A' = B \cap B' = \emptyset$, then $A \cup A' \sim_{(i+j)} B \cup B'$.

Proof. Part (e) follows from 2.10, and the rest follows directly from the definition of i-equidecomposability and 2.2. □

Lemma 2.14. *If a Borel set $A \subseteq X$ is i-compressible, then so is $[A]_G$. In fact, if $\mathcal{F}$ is a finite collection of Borel sets witnessing the i-compressibility of A, then it also witnesses that of $[A]_G$.*

Proof. Let $B \subseteq A$ be a Borel set such that $[A \setminus B]_G = [A]_G$ and $A \sim_i B$. Furthermore, let $\mathcal{I}$ be an A, B-sensitive partition generated by a collection $\mathcal{F}$ of i Borel sets such that $A \sim_{\mathcal{I}} B$. Let $\gamma : A \to G$ be a witnessing map for $A \sim_I B$. Put $A' = [A]_G$, $B' = B \cup (A' \setminus A)$ and note that A', B' respect $\mathcal{I}$. Define $\gamma' : A' \to G$ by setting $\gamma'\downharpoonright_{A' \setminus A} = id\downharpoonright_{A' \setminus A}$ and $\gamma'\downharpoonright_A = \gamma$. Since A' respects $\mathcal{I}$ and $id\downharpoonright_{A' \setminus A}, \gamma$ are $F_{\mathcal{I}}$-invariant, γ' is $F_{\mathcal{I}}$-invariant and thus clearly witnesses $A' \sim_{\mathcal{I}} B'$. □

The following is a technical refinement of the definition of i-compressibility that is (again) necessary for $\mathfrak{C}_i$, defined below, to be a σ-ideal.

Definition 2.15 (i^*-compressibility)**.** *For $i \geq 1$, we say that a Borel set A is i^*-compressible if there is a Borel set $B \subseteq A$ such that $[A \setminus B]_G = [A]_G =: P$, $A \sim_i B$, and the latter is witnessed by a collection $\mathcal{F}$ of Borel sets such that $B \in \mathcal{F}^P$.*

Finally, for $i \geq 1$, put

$$\mathfrak{C}_i = \{A \subseteq X : \text{there is a } G\text{-invariant Borel set } P \supseteq A \text{ such that } P \text{ is } i^*\text{-compressible}\}.$$

Lemma 2.16. *Let $i \geq 1$ and $A \subseteq X$ be Borel. If $A \prec_i A$, then $A \in \mathfrak{C}_{i+1}$.*

Proof. Setting $P = [A]_G$ and applying 2.14, we get that $P \prec_i P$, i.e. there is $B \subseteq P$ such that $[P \setminus B]_G = P$ and $P \sim_i B$. Let $\mathcal{F}$ be a collection of Borel sets witnessing the latter fact. Then $\mathcal{F}' = \mathcal{F} \cup \{B\}$ witnesses $P \sim_{(i+1)} B$ and contains B. □

Proposition 2.17. *For all $i \geq 1$, $\mathfrak{C}_i$ is a σ-ideal.*

Proof. We only need to show that $\mathfrak{C}_i$ is closed under countable unions. For this it is enough to show that if $A_n \in \mathfrak{B}(X)$ are i^*-compressible G-invariant Borel sets, then so is $A := \bigcup_{n \in \mathbb{N}} A_n$.

We may assume that A_n are pairwise disjoint since we could replace each A_n by $A_n \setminus (\bigcup_{k<n} A_k)$. Let $B_n \subseteq A_n$ be a Borel set and $\mathcal{F}_n = \{F_k^n\}_{k<i}$ be a collection of Borel sets with $(F_0^n)^{A_n} = B_n$ such that $\mathcal{F}_n$ witnesses $A_n \sim_i B_n$ and $[A_n \setminus B_n]_G = A_n$. Using part (b) of 2.13, we may assume that $\mathcal{F}_n^{A_n} = \mathcal{F}_n$; in particular, $F_0^n = B_n$.

Put $B = \bigcup_{n \in \mathbb{N}} B_n$ and $F_k = \bigcup_{n \in \mathbb{N}} F_k^n$, $\forall k < i$; note that $F_0 = B$. Set $\mathcal{F} = \{F_k\}_{k<i}$ and $\mathcal{I} = <\mathcal{F}>$. Since $B \in \mathcal{F}$ and A is G-invariant, $\mathcal{I}$ is A, B-sensitive. Furthermore, since $\mathcal{F}^{A_n} = \mathcal{F}_n$, $A_n \sim_{\mathcal{I}} B_n$ for all $n \in \mathbb{N}$. Thus, by 2.9, $A \sim_{\mathcal{I}} B$ and hence A is i^*-compressible. □

2.3. **Traveling sets**

Definition 2.18. *Let $A \in \Gamma(X)$.*

- *We call A a traveling set with Γ pieces if there exists pairwise disjoint sets $\{A_n\}_{n \in \mathbb{N}}$ in $\Gamma(X)$ such that $A_0 = A$ and $A \sim^\Gamma A_n$, $\forall n \in \mathbb{N}$.*
- *For a finite Borel partition $\mathcal{I}$, we say that A is $\mathcal{I}$-traveling with Γ pieces if A respects $\mathcal{I}$ and the above condition holds with $\sim^\Gamma$ replaced by $\sim_{\mathcal{I}}^\Gamma$.*
- *For $i \geq 1$, we say that A is i-traveling if it is $\mathcal{I}$-traveling for some A-sensitive partition $\mathcal{I}$ generated by a collection of i Borel sets.*

Definition 2.19. *For a set $A \subseteq X$, a function $\gamma : A \to G^{\mathbb{N}}$ is called a travel guide for A if $\forall x \in A, \gamma(x)(0) = 1_G$ and $\forall (x, n) \neq (y, m) \in A \times \mathbb{N}$, $\gamma(x)(n)x \neq \gamma(y)(m)y$.*

For $A \in \Gamma(X)$, a Γ-measurable map $\gamma : A \to G^{\mathbb{N}}$ and $n \in \mathbb{N}$, set $\gamma_n := \gamma(\cdot)(n) : A \to G$ and note that γ_n is also Γ-measurable.

Observation 2.20. *Suppose $A \in \Gamma(X)$ and $\mathcal{I}$ is an A-sensitive finite Borel partition of X. Then A is $\mathcal{I}$-traveling with Γ pieces if and only if it has a Γ-measurable $F_{\mathcal{I}}$-invariant travel guide.*

Proof. Follows from definitions and Proposition 2.5. □

Now we establish the connection between compressibility and traveling sets.

Lemma 2.21. *Let $\mathcal{I}$ be a finite Borel partition of X, $P \in \Gamma(X)$ be a Borel G-invariant set and let A, B be Γ subsets of P. If $P \sim_{\mathcal{I}}^\Gamma B$, then $P \setminus B$ is $\mathcal{I}$-traveling with Γ pieces. Conversely, if A is $\mathcal{I}$-traveling with Γ pieces, then $P \sim_{\mathcal{I}}^\Gamma (P \setminus A)$. The same is true if we replace $\sim_{\mathcal{I}}^\Gamma$ and "$\mathcal{I}$-traveling" with $\sim^\Gamma$ and "traveling", respectively.*

Proof. For the first statement, let $\gamma : X \to G$ be a witnessing map for $X \sim_{\mathcal{I}}^{\Gamma} B$. Put $A' = X \setminus B$ and note that A' respects $\mathcal{I}$ since so does P and hence B. We show that A' is $\mathcal{I}$-traveling. Put $A_n = (\hat{\gamma})^n(A')$, for each $n \geq 0$. It follows from injectivity of $\hat{\gamma}$ that A_n are pairwise disjoint. For all n, recursively define $\delta_n : A' \to G$ as follows

$$\begin{cases} \delta_0 = \gamma|_{A'} \\ \delta_{n+1} = \gamma \circ \hat{\delta}_n \end{cases}.$$

It follows from $F_{\mathcal{I}}$-invariance of γ that each δ_n is $F_{\mathcal{I}}$-invariant. It is also clear that $\hat{\delta}_n = (\hat{\gamma})^n$ and hence δ_n is a witnessing map for $A' \sim_{\mathcal{I}}^{\Gamma} A_n$. Thus A' is i-traveling with Γ pieces.

For the converse, assume that A is $\mathcal{I}$-traveling and let $\{A_n\}_{n\in\mathbb{N}}$ be as in Definition 2.18. In particular, each A_n respects $\mathcal{I}$ and $A_n \sim_{\mathcal{I}}^{\Gamma} A_m$, for all $n, m \in \mathbb{N}$. Let $P' = \bigcup_{n\in\mathbb{N}} A_n$ and $B' = \bigcup_{n\geq 1} A_n$. Since $A_n \sim_{\mathcal{I}}^{\Gamma} A_{n+1}$, part (b) of 2.2 implies that $P' \sim_{\mathcal{I}}^{\Gamma} B'$. Moreover, since $P \setminus P' \sim_{\mathcal{I}}^{\Gamma} P \setminus P'$, we get $P \sim_{\mathcal{I}}^{\Gamma} (B' \cup (P \setminus P')) = P \setminus A$. □

For a G-invariant set P and $A \subseteq P$, we say that A is a complete section for P if $[A]_G = P$. The above lemma immediately implies the following.

Proposition 2.22. *Let $P \in \Gamma(X)$ be G-invariant and $i \geq 1$. P is i-compressible with Γ pieces if and only if there exists a complete section for P that is i-traveling with Γ pieces. The same is true with "i-compressible" and "i-traveling" replaced by "compressible" and "traveling".*

We need the following lemma in the proofs of 2.24 and 2.25.

Lemma 2.23. *Suppose $A \subseteq X$ is an invariant analytic set that does not admit an invariant Borel probability measure. Then there is an invariant Borel set $A' \supseteq A$ that still does not admit an invariant Borel probability measure.*

Proof. Let $\mathcal{M}$ denote the standard Borel space of G-invariant Borel probability measures on X (see Section 17 in [Kec95]). Let $\Phi \subseteq Pow(X)$ be the following predicate:

$$\Phi(W) \Leftrightarrow \forall \mu \in \mathcal{M}(\mu(W) = 0).$$

Claim. *There is a Borel set $B \supseteq A$ with $\Phi(B)$.*

Proof of Claim. By the dual form of the First Reflection Theorem for $\mathbf{\Pi}_1^1$ (see the discussion following 35.10 in [Kec95]), it is enough to show that Φ is $\mathbf{\Pi}_1^1$ on $\mathbf{\Sigma}_1^1$. To this end, let Y be a Polish space and $D \subseteq Y \times X$ be analytic. Then, for any $n \in \mathbb{N}$, the set

$$H_n = \{(\mu, y) \in \mathcal{M} \times Y : \mu(D_y) > \frac{1}{n}\},$$

is analytic by a theorem of Kondô-Tugué (see 29.26 of [Kec95]), and hence so are the sets $H_n' := \mathrm{proj}_Y(H_n)$ and $H := \bigcup_{n\in\mathbb{N}} H_n'$. Finally, note that

$$\{y \in Y : \Phi(A_y)\} = \{y \in Y : \exists \mu \in \mathcal{M} \exists n \in \mathbb{N}(\mu(A_y) > \frac{1}{n})\}^c = H^c,$$

and so $\{y \in Y : \Phi(A_y)\}$ is $\mathbf{\Pi}_1^1$. ⊣

Now put $A' = (B)_G$, where $(B)_G = \{x \in B : [x]_G \subseteq B\}$. Clearly, A' is an invariant Borel set, $A' \supseteq A$, and $\Phi(A')$ since $A' \subseteq B$ and $\Phi(B)$. □

Proposition 2.24. *Let X be a Borel G-space. The following are equivalent:*

(1) X is compressible with universally measurable pieces;

(2) There is a universally measurable complete section that is a traveling set with universally measurable pieces;
(3) There is no G-invariant Borel probability measure on X;
(4) X is compressible with Borel pieces;
(5) There is a Borel complete section that is a traveling set with Borel pieces.

Proof. Equivalence of (1) and (2) as well as (4) and (5) is asserted in 2.22, (4)⇒(1) is trivial, and (3)⇒(4) follows from Nadkarni's theorem (see 1.16). It remains to show (1)⇒(3). To this end, suppose $X \sim^{\Gamma} B$, where $B^c = X \setminus B$ is a complete section and Γ is the class of universally measurable sets. If there was a G-invariant Borel probability measure μ on X, then $\mu(X) = \mu(B)$ and hence $\mu(B^c) = 0$. But since B^c is a complete section, $X = \bigcup_{g \in G} gB^c$, and thus $\mu(X) = 0$, a contradiction. □

Now we prove an analogue of this for i-compressibility.

Proposition 2.25. *Let X be a Borel G-space. For $i \geq 1$, the following are equivalent:*
(1) X is i-compressible with universally measurable pieces;
(2) There is a universally measurable complete section that is an i-traveling set with universally measurable pieces;
(3) There is a partition $\mathcal{I}$ of X generated by i Borel sets such that $Y = f_{\mathcal{I}}(X) \subseteq |\mathcal{I}|^G$ does not admit a G-invariant Borel probability measure;
(4) X is i-compressible with Borel pieces;
(5) There is a Borel complete section that is a i-traveling set with Borel pieces.

Proof. Equivalence of (1) and (2) as well as (4) and (5) is asserted in 2.22 and (4)⇒(1) is trivial. It remains to show (1)⇒(3)⇒(5).
(1)⇒(3): Suppose $X \sim^{\Gamma}_{\mathcal{I}} B$, where $B^c = X \setminus B$ is a complete section, $\mathcal{I}$ is a partition of X generated by i Borel sets, and Γ denotes the class of universally measurable sets. Let $\gamma : X \to G$ be a witnessing map for $X \sim^{\Gamma}_{i} B$. By the Jankov-von Neumann uniformization theorem (see 18.1 in [Kec95]), $f_{\mathcal{I}}$ has a $\sigma(\mathbf{\Sigma}^1_1)$-measurable (hence universally measurable) right inverse $h : Y \to X$. Define $\delta : Y \to G$ by $\delta(y) = \gamma(h(y))$ and note that δ is universally measurable being a composition of such functions. Letting $B' = \hat{\delta}(Y)$, it is straightforward to check that $\hat{\delta} \circ f_{\mathcal{I}} = f_{\mathcal{I}} \circ \hat{\gamma}$ and thus $B' = f_{\mathcal{I}}(\hat{\gamma}(X)) = f_{\mathcal{I}}(B)$. Now it follows that δ is a witnessing map for $Y \sim^{\Gamma} B'$ and hence Y is compressible with universally measurable pieces. Finally, (1)⇒(3) of 2.24 implies that Y does not admit an invariant Borel probability measure.
(3)⇒(5): Assume Y is as in (3). Then by Lemma 2.23, there is a Borel G-invariant $Y' \supseteq Y$ that does not admit a G-invariant Borel probability measure. Viewing Y' as a Borel G-space, we apply (3)⇒(4) of 2.24 and get that Y' is compressible with Borel pieces; thus there is a Borel $B' \subseteq Y'$ with $[Y' \setminus B']_G = Y'$ such that $Y' \sim B'$. Let $\delta : Y' \to G$ be a witnessing map for $Y' \sim B'$. Put $B = f_{\mathcal{I}}^{-1}(B')$ and $\gamma = \delta \circ f_{\mathcal{I}}$. By definition, γ is $F_{\mathcal{I}}$-invariant. In fact, it is straightforward to check that γ is a witnessing map for $X \sim_{\mathcal{I}} B$ and $[X \setminus B]_G = [f_{\mathcal{I}}^{-1}(Y \setminus B')]_G = f_{\mathcal{I}}^{-1}([Y \setminus B']_G) = f_{\mathcal{I}}^{-1}(Y) = X$. Hence X is $\mathcal{I}$-compressible. □

We now give an example of a 1-traveling set. First we need some definitions.

Definition 2.26. *Let X be a Borel G-space and $A \subseteq X$ be Borel. A is called*

- *aperiodic if it intersects every orbit in either 0 or infinitely many points;*
- *a partial transversal if it intersects every orbit in at most one point;*

- *smooth if there is a Borel partial transversal $T \subseteq A$ such that $[T]_G = [A]_G$.*

Proposition 2.27. *Let X be an aperiodic Borel G-space and $T \subseteq X$ be Borel. If T is a partial transversal, then T is $<T>$-traveling.*

Proof. let $G = \{g_n\}_{n\in\mathbb{N}}$ with $g_0 = 1_G$. For each $n \in \mathbb{N}$, define $\bar{n} : X \to \mathbb{N}$ and $\gamma_n : T \to G$ recursively in n as follows:

$$\begin{cases} \bar{n}(x) = \text{the least } k \text{ such that } g_k x \notin \{\hat{\gamma}_i(x) : i < n\} \\ \gamma_n(x) = g_{\bar{n}(x)} \end{cases}.$$

Clearly, $\bar{n}$ and γ_n are well-defined and Borel. Define $\gamma : T \to G^{\mathbb{N}}$ by setting $\gamma(\cdot)(n) = \gamma_n$. It follows from the definitions that γ is a Borel travel guide for T and hence, T is a traveling set. It remains to show that γ is $F_{\mathcal{I}}$-invariant, where $\mathcal{I} =<T>$. For this it is enough to show that $\bar{n}$ is $F_{\mathcal{I}}$-invariant, which we do by induction on n. Since it trivially holds for $n = 0$, we assume it is true for all $0 \le k < n$ and show it for n. To this end, suppose $x, y \in T$ with $xF_{\mathcal{I}}y$, and assume for contradiction that $m := \bar{n}(x) < \bar{n}(y)$. Thus it follows that $g_m y = \hat{\gamma}_k(y) \in \hat{\gamma}_k(T)$, for some $k < n$. By the induction hypothesis, $\hat{\gamma}_k(T)$ is $F_{\mathcal{I}}$-invariant and hence, $g_m x \in \hat{\gamma}_k(T)$, contradicting the definition of $\bar{n}(x)$. $\square$

Corollary 2.28. *Let X be an aperiodic Borel G-space. If a Borel set $A \subseteq X$ is smooth, then $A \in \mathfrak{C}_1$.*

Proof. Let $P = [A]_G$ and let T be a Borel partial transversal with $[T]_G = P$. By 2.27, T is $\mathcal{I}$-traveling, where $\mathcal{I} =<T>$. Hence, $P \sim_{\mathcal{I}} P \setminus T$, by Lemma 2.21. This implies that P is 1^*-compressible since $\mathcal{I} =<T^c>$ and $P \setminus T \in \{T^c\}^P$. $\square$

2.4. **Constructing finite generators using i-traveling sets**

Lemma 2.29. *Let $A \in \mathfrak{B}(X)$ be a complete section and $\mathcal{I}$ be an A-sensitive finite Borel partition of X. If A is $\mathcal{I}$-traveling (with Borel pieces), then there is a Borel $2|\mathcal{I}|$-generator. If moreover $A \in \mathcal{I}$, then there is a Borel $(2|\mathcal{I}| - 1)$-generator.*

Proof. Let γ be an $F_{\mathcal{I}}$-invariant Borel travel guide for A. Fix a countable family $\{U_n\}_{n\in\mathbb{N}}$ generating the Borel structure of X and let $B = \bigcup_{n\ge1} \hat{\gamma}_n(A \cap U_n)$. By Lemma 2.3, each $\hat{\gamma}_n$ maps Borel sets to Borel sets and hence B is Borel. Set $\mathcal{J} =<B>$, $\mathcal{P} = \mathcal{I} \vee \mathcal{J}$ and note that $|\mathcal{P}| \le 2|\mathcal{I}|$. A and B are disjoint since $\{\hat{\gamma}_n(A)\}_{n\in\mathbb{N}}$ is a collection of pairwise disjoint sets and $\hat{\gamma}_0(A) = A$; thus if $A \in \mathcal{I}$, $|\mathcal{P}| \le 1 + 2(|\mathcal{I}| - 1) = 2|\mathcal{I}| - 1$. We show that $\mathcal{P}$ is a generator, that is $G\mathcal{P}$ separates points in X.

Let $x \ne y \in X$ and assume they are not separated by $G\mathcal{I}$, thus $xF_{\mathcal{I}}y$. We show that $G\mathcal{J}$ separates x and y. Because A is a complete section, multiplying x by an appropriate group element, we may assume that $x \in A$. Since A respects $\mathcal{I}$, A is $F_{\mathcal{I}}$-invariant and thus $y \in A$. Also, because γ is $F_{\mathcal{I}}$-invariant, $\gamma_n(x) = \gamma_n(y)$, $\forall n \in \mathbb{N}$. Let $n \ge 1$ be such that $x \in U_n$ but $y \notin U_n$. Put $g = \gamma_n(x)(= \gamma_n(y))$. Then $gx = \hat{\gamma}_n(x) \in \hat{\gamma}_n(A \cap U_n)$ while $gy = \hat{\gamma}_n(y) \notin \hat{\gamma}_n(A \cap U_n)$. Hence, $gx \in B$ and $gy \notin B$ because $\gamma_m(A) \cap \gamma_n(A) = \emptyset$ for all $m \ne n$ and $gy = \hat{\gamma}_n(y) \in \hat{\gamma}_n(A)$. Thus $G\mathcal{J}$ separates x and y. $\square$

Now 2.25 and 2.29 together imply the following.

Proposition 2.30. *Let X be a Borel G-space and $i \ge 1$. If X is i-compressible then there is a Borel 2^{i+1}-generator.*

Proof. By 2.25, there exists a Borel i-traveling complete section A. Let $\mathcal{I}$ witness A being i-traveling and thus, by Lemma 2.29, there is a $2|\mathcal{I}| \leq 2 \cdot 2^i = 2^{i+1}$-generator. $\square$

Example 2.31. For $2 \leq n \leq \infty$, let $\mathbb{F}_n$ denote the free group on n generators and let X be the boundary of $\mathbb{F}_n$, i.e. the set of infinite reduced words. Clearly, the product topology makes X a Polish space and $\mathbb{F}_n$ acts continuously on X by left concatenation and cancellation. We show that X is 1-compressible and thus admits a Borel $2^2 = 4$-generator by Proposition 2.30. To this end, let a, b be two of the n generators of $\mathbb{F}_n$ and let X_a be the set of all words in X that start with a. Then $X = (X_{a^{-1}} \cup X_{a^{-1}}^c) \sim_{\mathcal{I}} Y$, where $Y = bX_{a^{-1}} \cup aX_{a^{-1}}^c$ and $\mathcal{I} < X_{a^{-1}} >$. Hence $X \sim_1 Y$. Since $X \setminus Y \supseteq X_{a^{-1}}$, $[X \setminus Y]_{\mathbb{F}_n} = X$ and thus X is 1-compressible.

Now we obtain a sufficient condition for the existence of an embedding into a finite Bernoulli shift.

Corollary 2.32. *Let X be a Borel G-space and $k \in \mathbb{N}$. If there exists a Borel G-map $f : X \to k^G$ such that $Y = f(X)$ does not admit a G-invariant Borel probability measure, then there is a Borel G-embedding of X into $(2k)^G$.*

Proof. Let $\mathcal{I} = \mathcal{I}_f$ and hence $f = f_{\mathcal{I}}$. By (3)⇒(5) of 2.25 (or rather the proof of it), X admits a Borel $\mathcal{I}$-traveling complete section. Thus by Lemma 2.29, X admits a $2|\mathcal{I}| = 2k$-generator and hence, there is a Borel G-embedding of X into $(2k)^G$. $\square$

Lemma 2.33. *Let $\mathcal{I}$ be a partition of X into n Borel sets. Then $\mathcal{I}$ is generated by $k = \lceil \log_2(n) \rceil$ Borel sets.*

Proof. Since $2^k \geq n$, we can index $\mathcal{I}$ by the set $\mathbf{2^k}$ of all k-tuples of $\{0, 1\}$, i.e. $\mathcal{I} = \{A_\sigma\}_{\sigma \in \mathbf{2^k}}$. For all $i < k$, put

$$B_i = \bigcup_{\sigma \in \mathbf{2^k} \wedge \sigma(i)=1} A_\sigma.$$

Now it is clear that for all $\sigma \in \mathbf{2^k}$, $A_\sigma = \bigcap_{i<k} B_i^{\sigma(i)}$, where $B_i^{\sigma(i)}$ is equal to B_i if $\sigma(i) = 1$, and equal to B_i^c, otherwise. Thus $\mathcal{I} = <B_i : i < k>$. $\square$

Proposition 2.34. *If X is compressible and there is a Borel n-generator, then X is $\lceil \log_2(n) \rceil$-compressible.*

Proof. Let $\mathcal{I}$ be an n-generator and hence, by Lemma 2.33, $\mathcal{I}$ is generated by i Borel sets. Since $G\mathcal{I}$ separates points in X, each $F_{\mathcal{I}}$-class is a singleton and hence $X \prec X$ implies $X \prec_{\mathcal{I}} X$. $\square$

From 2.30 and 2.34 we immediately get the following corollary, which justifies the use of i-compressibility in studying Question 1.6.

Corollary 2.35. *Let X be a Borel G-space that is compressible (equivalently, does not admit an invariant Borel probability measure). X admits a finite generator if and only if X is i-compressible for some $i \geq 1$.*

3. Invariant measures and i-compressibility

This section is mainly devoted to proving the following theorem.

Theorem 3.1. *Let X be a Borel G-space. If X is aperiodic, then there exists a function $m : \mathfrak{B}(X) \times X \to [0,1]$ satisfying the following properties for all $A, B \in \mathfrak{B}(X)$:*
(a) $m(A, \cdot)$ is Borel;
(b) $m(X, x) = 1$, $\forall x \in X$;
(c) If $A \subseteq B$, then $m(A, x) \le m(B, x)$, $\forall x \in X$;
(d) $m(A, x) = 0$ off $[A]_G$;
(e) $m(A, x) > 0$ on $[A]_G$ modulo $\mathfrak{C}_4$;
(f) $m(A, x) = m(gA, x)$, for all $g \in G$, $x \in X$ modulo $\mathfrak{C}_3$;
(g) If $A \cap B = \emptyset$, then $m(A \cup B, x) = m(A, x) + m(B, x)$, $\forall x \in X$ modulo $\mathfrak{C}_4$.

Remark. A version of this theorem is what lies at the heart of the proof of Nadkarni's theorem. The conclusions of our theorem are modulo $\mathfrak{C}_4$, which is potentially a smaller σ-ideal than the σ-ideal of sets contained in compressible Borel sets used in Nadkarni's version. However, the price we paid for this is that part (g) asserts only finite additivity instead of countable additivity asserted by Nadkarni's version.

Proof of Theorem 3.1. Our proof follows the general outline of Nadkarni's proof. The construction of $m(A, x)$ is somewhat similar to that of Haar measure. First, for sets A, B, we define a Borel function $[A/B] : X \to \mathbb{N} \cup \{-1, \infty\}$ that basically gives the number of copies of $B^{[x]_G}$ that fit in $A^{[x]_G}$ when moved by group elements (piecewise). Then we define a decreasing sequence of complete sections (called a fundamental sequence below), which serves as a gauge to measure the size of a given set.

Assume throughout that X is an aperiodic Borel G-space (although we only use the aperiodicity assumption in 3.9 to assert that smooth sets are in $\mathfrak{C}_1$).

Lemma 3.2 (Comparability)**.** *$\forall A, B \in \mathfrak{B}(X)$, there is a partition $X = P \cup Q$ into G-invariant Borel sets such that for any A, B-sensitive finite Borel partition $\mathcal{I}$ of X, $A^P \prec_{\mathcal{I}} B^P$ and $B^Q \preceq_{\mathcal{I}} A^Q$.*

Proof. It is enough to prove the lemma assuming $X = [A]_G \cap [B]_G$ since we can always include $[B]_G \setminus [A]_G$ in P and $X \setminus [B]_G$ in Q.

Fix an enumeration $\{g_n\}_{n\in\mathbb{N}}$ for G. We recursively construct Borel sets A_n, B_n, A'_n, B'_n as follows. Set $A'_0 = A$ and $B'_0 = B$. Assuming A'_n, B'_n are defined, set $B_n = B'_n \cap g_n A'_n$, $A_n = g_n^{-1} B_n$, $A'_{n+1} = A'_n \setminus A_n$ and $B'_{n+1} = B'_n \setminus B_n$.

It is easy to see by induction on n that for any A, B-sensitive $\mathcal{I}$, A_n, B_n are $F_{\mathcal{I}}$-invariant since so are A, B. Thus, setting $A^* = \bigcup_{n\in\mathbb{N}} A_n$ and $B^* = \bigcup_{n\in\mathbb{N}} B_n$, we get that $A^* \sim_{\mathcal{I}} B^*$ since $B_n = g_n A_n$.

Let $A' = A \setminus A^*$, $B' = B \setminus B^*$ and set $P = [B']_G$, $Q = X \setminus P$.

Claim. $[A']_G \cap [B']_G = \emptyset$.

Proof of Claim. Assume for contradiction that $\exists x \in A'$ and $n \in \mathbb{N}$ such that $g_n x \in B'$. It is clear that $A' = \bigcap_{k\in\mathbb{N}} A'_k$, $B' = \bigcap_{k\in\mathbb{N}} B'_k$; in particular, $x \in A'_n$ and $g_n x \in B'_n$. But then $g_n x \in B_n$ and $x \in A_n$, contradicting $x \in A'$. ⊣

Let $\mathcal{I}$ be an A, B-sensitive partition. Then $A^P = (A^*)^P$ and hence $A^P \prec_{\mathcal{I}} B^P$ since $(A^*)^P \sim_{\mathcal{I}} (B^*)^P \subseteq B^P$ and $[B^P \setminus (B^*)^P]_G = [B']_G = P = [B^P]_G$. Similarly, $B^Q = (B^*)^Q$ and hence $B^Q \preceq_{\mathcal{I}} A^Q$ since $(B^*)^Q \sim_{\mathcal{I}} (A^*)^Q \subseteq A^Q$. □

Definition 3.3 (Divisibility)**.** *Let $n \leq \infty$, $A, B, C \in \mathfrak{B}(X)$ and $\mathcal{I}$ be a finite Borel partition of X.*

- *Write $A \sim_{\mathcal{I}} nB \oplus C$ if there are Borel sets $A_k \subseteq A$, $k < n$, such that $\{A_k\}_{k<n} \cup \{C\}$ is a partition of A, each A_k is $F_{\mathcal{I}}$-invariant relative to A and $A_k \sim_{\mathcal{I}} B$.*
- *Write $nB \preceq_{\mathcal{I}} A$ if there is $C \subseteq A$ with $A \sim_{\mathcal{I}} nB \oplus C$, and write $nB \prec_{\mathcal{I}} A$ if moreover $[C]_G = [A]_G$.*
- *Write $A \preceq_{\mathcal{I}} nB$ if there is a Borel partition $\{A_k\}_{k<n}$ of A such that each A_k is $F_{\mathcal{I}}$-invariant relative to A and $A_k \preceq_{\mathcal{I}} B$. If moreover, $A_k \prec_{\mathcal{I}} B$ for at least one $k < n$, we write $A \prec_{\mathcal{I}} nB$.*

For $i \geq 1$, we use the above notation with $\mathcal{I}$ replaced by i if there is an A, B-sensitive partition $\mathcal{I}$ generated by i sets for which the above conditions hold.

Proposition 3.4 (Euclidean decomposition)**.** *Let $A, B \in \mathfrak{B}(X)$ and put $R = [A]_G \cap [B]_G$. There exists a partition $\{P_n\}_{n\leq\infty}$ of R into G-invariant Borel sets such that for any A, B-sensitive finite Borel partition $\mathcal{I}$ of X and $n \leq \infty$, $A^{P_n} \sim_{\mathcal{I}} nB^{P_n} \oplus C_n$ for some C_n such that $C_n \prec_{\mathcal{I}} B^{P_n}$, if $n < \infty$.*

Proof. We repeatedly apply Lemma 3.2. For $n < \infty$, recursively define R_n, P_n, A_n, C_n satisfying the following:

(i) R_n are invariant decreasing Borel sets such that $nB^{R_n} \preceq_{\mathcal{I}} A^{R_n}$ for any A, B-sensitive $\mathcal{I}$;
(ii) $P_n = R_n \setminus R_{n+1}$;
(iii) $A_n \subseteq R_{n+1}$ are pairwise disjoint Borel sets such that for any A, B-sensitive $\mathcal{I}$, every A_n respects $\mathcal{I}$ and $A_n \sim_{\mathcal{I}} B^{R_{n+1}}$;
(iv) $C_n \subseteq P_n$ are Borel sets such that for any A, B-sensitive $\mathcal{I}$, every C_n respects $\mathcal{I}$ and $C_n \prec_{\mathcal{I}} B^{P_n}$.

Set $R_0 = R$. Given R_n, $\{A_k\}_{k<n}$ satisfying the above properties, let $A' = A^{R_n} \setminus \bigcup_{k<n} A_k$. We apply Lemma 3.2 to A' and B^{R_n}, and get a partition $R_n = P_n \cup R_{n+1}$ such that $(A')^{P_n} \prec_{\mathcal{I}} B^{P_n}$ and $B^{R_{n+1}} \preceq_{\mathcal{I}} (A')^{R_{n+1}}$. Set $C_n = (A')^{P_n}$. Let $A_n \subseteq (A')^{R_{n+1}}$ be such that $B^{R_{n+1}} \sim_{\mathcal{I}} A_n$. It is straightforward to check (i)-(iv) are satisfied.

Now let $R_\infty = \bigcap_{n\in\mathbb{N}} R_n$ and $C_\infty = (A \setminus \bigcup_{n\in\mathbb{N}} A_n)^{R_\omega}$. Now it follows from (i)-(iv) that for all $n \leq \infty$, $\{A_k^{P_n}\}_{k<n} \cup \{C_n\}$ is a partition of A^{P_n} witnessing $A^{P_n} \sim_{\mathcal{I}} nB \oplus C_n$, and for all $n < \infty$, $C_n \prec B^{P_n}$. □

For $A, B \in \mathfrak{B}(X)$, let $\{P_n\}_{n\leq\infty}$ be as in the above proposition. Define

$$[A/B](x) = \begin{cases} n & \text{if } x \in P_n, n < \infty \\ \infty & \text{if } x \in P_\infty \text{ or } x \in [A]_G \setminus [B]_G \\ 0 & \text{if } x \in [B]_G \setminus [A]_G \\ -1 & \text{otherwise} \end{cases}.$$

Note that $[A/B] : X \to \mathbb{N} \cup \{-1, \infty\}$ is a Borel function by definition.

Lemma 3.5 (Infinite divisibility $\Rightarrow$ compressibility)**.** *Let $A, B \in \mathfrak{B}(X)$ with $[A]_G = [B]_G$, and let $\mathcal{I}$ be a finite Borel partition of X. If $\infty B \preceq_{\mathcal{I}} A$, then $A \prec_{\mathcal{I}} A$.*

Proof. Let $C \subseteq A$ be such that $A \sim_{\mathcal{I}} \infty B \oplus C$ and let $\{A_k\}_{k<\infty}$ be as in Definition 3.3. $A_k \sim_{\mathcal{I}} B \sim_{\mathcal{I}} A_{k+1}$ and hence $A_k \sim_{\mathcal{I}} A_{k+1}$. Also trivially $C \sim_{\mathcal{I}} C$. Thus, letting $A' = \bigcup_{k<\infty} A_{k+1} \cup C$, we apply (b) of 2.2 to A and A', and get that $A \sim_{\mathcal{I}} A'$. Because $[A \setminus A']_G = [A_0]_G = [B]_G = [A]_G$, we have $A \prec_{\mathcal{I}} A$. □

Lemma 3.6 (Ambiguity $\Rightarrow$ compressibility)**.** *Let $A, B \in \mathfrak{B}(X)$ and $\mathcal{I}$ be a finite Borel partition of X. If $nB \preceq_{\mathcal{I}} A \prec_{\mathcal{I}} nB$ for some $n \geq 1$, then $A \prec_{\mathcal{I}} A$.*

Proof. Let $C \subseteq A$ be such that $A \sim_{\mathcal{I}} nB \oplus C$ and let $\{A_k\}_{k<n}$ be a partitions of $A \setminus C$ witnessing $A \sim_{\mathcal{I}} nB \oplus C$. Also let $\{A'_k\}_{k<n}$ be witnessing $A \prec_{\mathcal{I}} nB$ with $A'_0 \prec_{\mathcal{I}} B$. Since $A'_k \preceq_{\mathcal{I}} B \sim_{\mathcal{I}} A_k$, $A'_k \preceq_{\mathcal{I}} A_k$, for all $k < n$ and $A'_0 \prec_{\mathcal{I}} A_0$. Note that it follows from the hypothesis that $[A]_G = [B]_G$ and hence $[A_0]_G = [A]_G$ since $[A_0]_G = [B]_G$. Thus it follows from (b) of 2.2 that $A = \bigcup_{k<n} A'_k \prec_{\mathcal{I}} \bigcup_{k<n} A_k \subseteq A$. $\square$

Proposition 3.7. *Let $n \in \mathbb{N}$ and $A, A', B, P \in \mathfrak{B}(X)$, where P is invariant.*
(a) $[A/B] \in \mathbb{N}$ on $[B]_G$ modulo $\mathfrak{C}_3$.
(b) If $A \subseteq A'$, then $[A/B] \leq [A'/B]$.
(c) If $[A/B] = n$ on P then $nB^P \preceq_{\mathcal{I}} A^P \prec_{\mathcal{I}} (n+1)B^P$, for any finite Borel partition $\mathcal{I}$ that is A, B-sensitive. In particular, $nB^P \preceq_2 A^P \prec_2 (n+1)B^P$ by taking $\mathcal{I} =<A, B>$.
(d) For $n \geq 1$, if $A^P \prec_i nB^P$, then $[A/B] < n$ on P modulo $\mathfrak{C}_{i+1}$;
(e) If $A^P \subseteq [B]_G$ and $nB^P \preceq_i A^P$, then $[A/B] \geq n$ on P modulo $\mathfrak{C}_{i+1}$.

Proof. For (a), notice that 3.5 and 2.16 imply that $P_\infty \in \mathfrak{C}_3$. (b) and (c) follow from the definition of $[A/B]$. For (d), let $\mathcal{I}$ be an A, B-sensitive partition of X generated by i Borel sets such that $A^P \prec_{\mathcal{I}} nB^P$, and put $Q = \{x \in P : [A/B](x) \geq n\}$. By (c), $nB^Q \preceq_{\mathcal{I}} A^Q$. Thus, by Lemma 3.6, $A^Q \prec_{\mathcal{I}} A^Q$ and hence, by Lemma 2.16, $[A^Q]_G = Q \in C_{i+1}$.

For (e), let $\mathcal{I}$ be an A, B-sensitive partition of X generated by i Borel sets such that $nB^P \preceq_{\mathcal{I}} A^P$, and put $Q = \{x \in P : [A/B](x) < n\}$. By (c), $A^Q \prec_{\mathcal{I}} nB^Q$. Thus, by Lemma 3.6, $A^Q \prec_{\mathcal{I}} A^Q$ and hence, by Lemma 2.16, $[A^Q]_G = Q \in C_{i+1}$. $\square$

Definition 3.8 (Fundamental sequence)**.** *A sequence $\{F_n\}_{n\in\mathbb{N}}$ of decreasing Borel complete sections with $F_0 = X$ and $[F_n/F_{n+1}] \geq 2$ modulo $\mathfrak{C}_3$ is called fundamental.*

Proposition 3.9. *There exists a fundamental sequence.*

Proof. Take $F_0 = X$. Given any complete Borel section F, its intersection with every orbit is infinite modulo a smooth set (if the intersection of an orbit with a set is finite, then we can choose an element from each such nonempty intersection in a Borel way and get a Borel transversal). Thus, by 2.28, F is aperiodic modulo $\mathfrak{C}_1$. Now use Lemma 6.1 to write $F = A \cup B, A \cap B = \emptyset$, where A, B are also complete sections. Let now P, Q be as in Lemma 3.2 for A, B, and hence $A^P \prec_2 B^P, B^Q \preceq_2 A^Q$ because we can take $\mathcal{I} =<A, B>$. Let $A' = A^P \cup B^Q, B' = B^P \cup A^Q$. Then $F = A' \cup B', A' \cap B' = \emptyset$, $A' \preceq B'$ and A' is also a complete Borel section. By (e) of 3.7, $[F/A'] \geq 2$ modulo $\mathfrak{C}_3$. Iterate this process to inductively define F_n. $\square$

Fix a fundamental sequence $\{F_n\}_{n\in\mathbb{N}}$ and for any $A \in \mathfrak{B}(X), x \in X$, define

$$m(A, x) = \lim_{n\to\infty} \frac{[A/F_n](x)}{[X/F_n](x)}, \tag{†}$$

if the limit exists, and 0 otherwise. In the above fraction we define $\frac{\infty}{\infty} = 1$. We will prove in Proposition 3.12 that this limit exists modulo $\mathfrak{C}_4$. But first we need the following two lemmas.

Lemma 3.10 (Almost cancelation)**.** *For any $A, B, C \in X$,*

$$[A/B][B/C] \leq [A/C] < ([A/B] + 1)([B/C] + 1)$$

on $R := [B]_G \cap [C]_G$ modulo $\mathfrak{C}_4$.

Proof. Let $\mathcal{I} = <A, B, C>$.

$[A/B][B/C] \leq [A/C]$: Fix integers $i, j > 0$ and let $P = \{x \in X : [A/B](x) = i \wedge [B/C](x) = j\}$. Since $i, j > 0$, $P \subseteq [A]_G \cap [B]_G \cap [C]_G$ and we work in P. By (c) of 3.7, $iB \preceq_{\mathcal{I}} A$ and $jC \preceq_{\mathcal{I}} B$. Thus it follows that $ijC \preceq_{\mathcal{I}} A$ and hence $[A/C] \geq ij$ modulo $\mathfrak{C}_4$ by (e) of 3.7.

$[A/C] < ([A/B]+1)([B/C]+1)$: By (a) of 3.7, $[A/C], [A/B], [B/C] \in \mathbb{N}$ on R modulo $\mathfrak{C}_3$. Fix $i, j \in \mathbb{N}$ and let $Q = \{x \in R : [A/B](x) = i \wedge [B/C](x) = j\}$. We work in Q. By (c) of 3.7, $A \prec_{\mathcal{I}} (i+1)B$ and $B \prec_{\mathcal{I}} (j+1)C$. Thus $A \prec_{\mathcal{I}} (i+1)(j+1)C$ and hence $[A/C] < (i+1)(j+1)$ modulo $\mathfrak{C}_4$ by (d) of 3.7. $\square$

Lemma 3.11. *For any $A \in \mathfrak{B}(A)$,*

$$\lim_{n\to\infty}[A/F_n] = \begin{cases} \infty & on\ [A]_G \\ 0 & on\ X \setminus [A]_G \end{cases},\ modulo\ \mathfrak{C}_4.$$

Proof. The part about $X \setminus [A]_E$ is clear, so work in $[A]_E$, i.e. assume $X = [A]_G$. By (a) of 3.7 and 3.10, we have

$$\infty > [F_1/A] \geq [F_1/F_n][F_n/A] \geq 2^{n-1}[F_n/A],\ \text{modulo } \mathfrak{C}_4,$$

which holds for all n at once since $\mathfrak{C}_4$ is a σ-ideal. Thus $[F_n/A] \to 0$ modulo $\mathfrak{C}_4$ and hence, as $[F_n/A] \in \mathbb{N}$, $[F_n/A]$ is eventually 0, modulo $\mathfrak{C}_4$. So if

$$B_k := \{x \in [A]_G : [F/A](x) = 0\},$$

then $B_k \nearrow X$, modulo $\mathfrak{C}_4$. Now it follows from Lemma 3.2 that $[A/F_k] > 0$ on B_k modulo $\mathfrak{C}_4$. But

$$[A/F_{k+n}] \geq [A/F_k][F_k/F_{k+n}] \geq 2^n[A/F_k],\ \text{modulo } \mathfrak{C}_4,$$

so for every k, $[A/F_n] \to \infty$ on B_k modulo $\mathfrak{C}_4$. Since $B_k \nearrow X$ modulo $\mathfrak{C}_4$, we have $[A/F_n] \to \infty$ on X, modulo $\mathfrak{C}_4$. $\square$

Proposition 3.12. *For any Borel set $A \subseteq X$, the limit in (†) exists and is positive on $[A]_G$, modulo $\mathfrak{C}_4$.*

Proof.

Claim. *Suppose $B, C \in \mathfrak{B}(X)$, $i \in \mathbb{N}$ and $D_i = \{x \in X : [C/F_i](x) > 0\}$. Then*

$$\overline{\lim}\frac{[B/F_n]}{[C/F_n]} \leq \frac{[B/F_i]+1}{[C/F_i]}$$

on D_i, modulo $\mathfrak{C}_4$.

Proof of Claim. Working in D_i and using Lemma 3.10, $\forall j$ we have (modulo $\mathfrak{C}_4$)

$$\begin{aligned}
[B/F_{i+j}] &\leq ([B/F_i]+1)([F_i/F_{i+j}]+1) \\
[C/F_{i+j}] &\geq [C/F_i][F_i/F_{i+j}] > 0,
\end{aligned}$$

so

$$\begin{aligned}
\frac{[B/F_{i+j}]}{[C/F_{i+j}]} &\leq \frac{[B/F_i]+1}{[C/F_i]} \cdot \frac{[F_i/F_{i+j}]+1}{[F_i/F_{i+j}]} \\
&\leq \frac{[B/F_i]+1}{[C/F_i]} \cdot (1 + \frac{1}{2^j}),
\end{aligned}$$

from which the claim follows. $\dashv$

Applying the claim to $B = A$ and $C = X$ (hence $D_i = X$), we get that for all $i \in \mathbb{N}$

$$\overline{\lim_{n\to\infty}} \frac{[A/F_n](x)}{[X/F_n](x)} \leq \frac{[A/F_i](x) + 1}{[X/F_i](x)} (\text{modulo } \mathfrak{C}_4).$$

Thus

$$\overline{\lim_{n\to\infty}} \frac{[A/F_n]}{[X/F_n]} \leq \underline{\lim_{i\to\infty}} \frac{[A/F_i] + 1}{[X/F_i]} = \underline{\lim_{i\to\infty}} \frac{[A/F_i]}{[X/F_i]}$$

since $\lim_{i\to\infty} \frac{1}{[X/F_i]} = 0$.

To see that $m(A, x)$ is positive on $[A]_E$ modulo $\mathfrak{C}_4$ we argue as follows. We work in $[A]_G$. Applying the above claim to $B = X$ and $C = A$, we get

$$\frac{1}{m(A, x)} = \lim_{n\to\infty} \frac{[X/F_n]}{[A/F_n]} \leq \frac{[X/F_i] + 1}{[A/F_i]} < \infty \text{ on } D_i \text{ (modulo } \mathfrak{C}_4).$$

Thus $m(A, x) > 0$ on $\cup_{i\in\mathbb{N}} D_i$, modulo $\mathfrak{C}_4$. But $D_i \nearrow [A]_G$ because $[A/F_i] \to \infty$ as $i \to \infty$, and hence $m(A, x) > 0$ on $[A]_G$ modulo $\mathfrak{C}_4$. □

Lemma 3.13 (Invariance)**.** *For* $A, F \in \mathfrak{B}(X)$, $\forall g \in G, [A/F] = [gA/F]$, *modulo* $\mathfrak{C}_3$.

Proof. We may assume that $X = [A]_G \cap [F]_G$. Fix $g \in G$, $n \in \mathbb{N}$, and put $Q = \{x \in X : [gA/F](x) = n\}$. We work in Q. Let $\mathcal{I} =<A, F>$ and hence A, gA, F respect $\mathcal{I}$. By (c) of 3.7, $nF \preceq_{\mathcal{I}} gA$. But clearly $gA \sim_{\mathcal{I}} A$ and hence $nF \preceq_{\mathcal{I}} A$. Thus, by (e) of 3.7, $[A/F] \geq n = [gA/F]$, modulo $\mathfrak{C}_3$. By symmetry, $[gA/F] \geq [A/F]$ (modulo $\mathfrak{C}_3$) and the lemma follows. □

Lemma 3.14 (Almost additivity)**.** *For any* $A, B, F \in X$ *with* $A \cap B = \emptyset$, $[A/F] + [B/F] \leq [A \cup B/F] \leq [A/F] + [B/F] + 1$ *modulo* $\mathfrak{C}_4$.

Proof. Let $\mathcal{I} =<A, B, F>$.

$[A/F] + [B/F] \leq [A \cap B/F]$: Fix $i, j \in \mathbb{N}$ not both 0, say $i > 0$, and let $S = \{x \in X : [A/F](x) = i \wedge [B/F](x) = j\}$. Since $i > 0$, $S \subseteq [A]_G \cap [F]_G$ and we work in S. By (c) of 3.7, $iF^S \preceq_{\mathcal{I}} A^S$ and $jF^S \preceq_{\mathcal{I}} B^S$. Hence $(i + j)F^S \preceq_{\mathcal{I}} (A \cup B)^S$ and thus, by (e) of 3.7, $[A \cup B/F] \geq i + j$, modulo $\mathfrak{C}_4$.

$[A \cap B/F] \leq [A/F] + [B/F] + 1$: Outside $[F]_G$, the inequality clearly holds. Fix $i, j \in \mathbb{N}$ and let $M = \{x \in [F]_G : [A/F](x) = i \wedge [B/F](x) = j\}$. We work in M. By (c) of 3.7, $A \prec_{\mathcal{I}} (i + 1)F$ and $B \prec_{\mathcal{I}} (j + 1)F$. Thus it is clear that $A \cup B \prec_{\mathcal{I}} (i + j + 2)F$ and hence $[A \cup B/F] < i + j + 2$, modulo $\mathfrak{C}_4$, by (d) of 3.7. □

Now we are ready to finish the proof of Theorem 3.1. Fix $A, B \in \mathfrak{B}(X)$. The fact that $m(A, x) \in [0, 1]$ and parts (b) and (d) follow directly from the definition of $m(A, x)$. Part (a) follows from the fact that $[A/F_n]$ is Borel for all $n \in \mathbb{N}$. (c) follows from (b) of Lemma 3.7, and (e) and (f) are asserted by 3.12 and 3.13, respectively.

To show (g), we argue as follows. By Lemma 3.14, $[A/F_n] + [B/F_n] \leq [A \cup B/F_n] \leq [A/F_n] + [B/F_n] + 1$, modulo $\mathfrak{C}_4$, and thus

$$\frac{[A/F_n]}{[X/F_n]} + \frac{[B/F_n]}{[X/F_n]} \leq \frac{[A \cup B/F_n]}{[X/F_n]} \leq \frac{[A/F_n]}{[X/F_n]} + \frac{[B/F_n]}{[X/F_n]} + \frac{1}{[X/F_n]},$$

for all n at once, modulo $\mathfrak{C}_4$ (using the fact that $\mathfrak{C}_4$ is a σ-ideal). Since $[X/F_n] \geq 2^n$, passing to the limit in the inequalities above, we get $m(A, x) + m(B, x) \leq m(A \cup B, x) \leq m(A, x) + m(B, x)$. QED (Thm 3.1)

Theorem 3.1 will only be used via Corollary 3.16 and to state it we need the following.

Definition 3.15. *Let X be a Borel G-space. $\mathcal{B} \subseteq \mathfrak{B}(X)$ is called a Boolean G-algebra, if it is a Boolean algebra, i.e. is closed under finite unions and complements, and is closed under the G-action, i.e. $G\mathcal{B} = \mathcal{B}$.*

Corollary 3.16. *Let X be a Borel G-space and let $\mathcal{B} \subseteq \mathfrak{B}(X)$ be a countable Boolean G-algebra. For any $A \in \mathcal{B}$ with $A \notin \mathfrak{C}_4$, there exists a G-invariant finitely additive probability measure μ on $\mathcal{B}$ with $\mu(A) > 0$. Moreover, μ can be taken such that there is $x \in A$ such that $\forall B \in \mathcal{B}$ with $B \cap [x]_G = \emptyset$, $\mu(B) = 0$.*

Proof. Let $A \in \mathcal{B}$ be such that $A \notin \mathfrak{C}_4$. We may assume that $X = [A]_G$ by setting the (to be constructed) measure to be 0 outside $[A]_G$.

If X is not aperiodic, then by assigning equal point masses to the points of a finite orbit, we will have a probability measure on all of $\mathfrak{B}(X)$, so assume X is aperiodic.

Since $\mathfrak{C}_4$ is a σ-ideal and $\mathcal{B}$ is countable, Theorem 3.1 implies that there is a $P \in \mathfrak{C}_4$ such that (a)-(g) of the same theorem hold on $X \setminus P$ for all $A, B \in \mathcal{B}$. Since $A \notin \mathfrak{C}_4$, there exists $x_A \in A \setminus P$. Hence, letting $\mu(B) = m(B, x_A)$ for all $B \in \mathcal{B}$, conditions (b),(f) and (g) imply that μ is a G-invariant finitely additive probability measure on $\mathcal{B}$. Moreover, since $x_A \in [A]_G \setminus P$, $\mu(A) = m(A, x_A) > 0$. Finally, the last assertion follows from condition (d). □

Corollary 3.17. *Let X be a Borel G-space. For every Borel set $A \subseteq X$ with $A \notin \mathfrak{C}_4$, there exists a G-invariant finitely additive Borel probability measure μ (defined on all Borel sets) with $\mu(A) > 0$.*

Proof. The statement follows from 3.16 and a standard application of the Compactness Theorem of propositional logic. Here are the details.

We fix the following set of propositional variables

$$\mathcal{P} = \{P_{A,r} : A \in \mathfrak{B}(X), r \in [0,1]\},$$

with the following interpretation in mind:

$$P_{A,r} \Leftrightarrow \text{“the measure of } A \text{ is } \geq r\text{”}.$$

Define the theory T as the following set of sentences: for each $A, B \in \mathfrak{B}(X)$, $r, s \in [0,1]$ and $g \in G$,

(i) “$P_{A,0}$”$\in T$;
(ii) if $r > 0$, then “$\neg P_{\emptyset,r}$”$\in T$;
(iii) if $s \geq r$, then “$P_{A,s} \to P_{A,r}$”$\in T$;
(iv) if $A \cap B = \emptyset$, then “$(P_{A,r} \wedge P_{B,s}) \to P_{A\cup B,r+s}$”, “$(\neg P_{A,r} \wedge \neg P_{B,s}) \to \neg P_{A\cup B,r+s}$”$\in T$;
(v) “$P_{X,1}$”$\in T$;
(vi) “$P_{A,r} \to P_{gA,r}$”$\in T$.

If there is an assignment of the variables in $\mathcal{P}$ satisfying T, then for each $A \in \mathfrak{B}(X)$, we can define

$$\mu(A) = \sup\{r \in [0,1] : P_{A,r}\}.$$

Note that due to (i), μ is well defined for all $A \in \mathfrak{B}(X)$. In fact, it is straightforward to check that μ is a finitely additive G-invariant probability measure. Thus, we only need to show that T is satisfiable, for which it is enough to check that T is finitely satisfiable, by the Compactness Theorem of propositional logic (or by Tychonoff's theorem).

Let $T_0 \subseteq T$ be finite and let $\mathcal{P}_0$ be the set of propositional variables that appear in the sentences in T_0. Let $\mathcal{B}$ denote the Boolean G-algebra generated by the sets that appear in the indices of the variables in $\mathcal{P}_0$. By 3.16, there is a finitely additive G-invariant probability measure μ defined on $\mathcal{B}$. Consider the following assignment of the variables in $\mathcal{P}_0$: for all $P_{A,r} \in \mathcal{P}_0$,

$$P_{A,r} :\Leftrightarrow \mu(A) \geq r.$$

It is straightforward to check that this assignment satisfies T_0, and hence, T is finitely satisfiable. □

4. Finite generators in the case of σ-compact spaces

In this section we prove that the answer to Question 1.6 is positive in case X has a σ-compact realization. To do this, we first prove Proposition 4.2, which shows how to construct a countably additive invariant probability measure on X using a finitely additive one. We then use 3.16 to conclude the result.

For the next two statements, let X be a second countable Hausdorff topological space equipped with a continuous action of G.

Lemma 4.1. *Let $\mathcal{U} \subseteq Pow(X)$ be a countable base for X closed under the G-action and finite unions/intersections. Let ρ be a G-invariant finitely additive probability measure on the G-algebra generated by $\mathcal{U}$. For every $A \subseteq X$, define*

$$\mu^*(A) = \inf\{\sum_{n\in\mathbb{N}} \rho(U_n) : U_n \in \mathcal{U} \ \wedge \ A \subseteq \bigcup_{n\in\mathbb{N}} U_n\}.$$

Then:

(a) μ^* is a G-invariant outer measure.
(b) If $K \subseteq X$ is compact, then K is metrizable and μ^* is a metric outer measure on K (with respect to any compatible metric).

Proof. It is a standard fact from measure theory that μ^* is an outer measure. That μ^* is G-invariant follows immediately from G-invariance of ρ and the fact that $\mathcal{U}$ is closed under the action of G.

For (b), first note that by Urysohn metrization theorem, K is metrizable, and fix a metric on K. If $E, F \subseteq K$ are a positive distance apart, then so are $\bar{E}$ and $\bar{F}$. Hence there exist disjoint open sets U, V such that $\bar{E} \subseteq U$, $\bar{F} \subseteq V$. Because $\bar{E}$ and $\bar{F}$ are compact, U, V can be taken to be finite unions of sets in $\mathcal{U}$ and therefore $U, V \in \mathcal{U}$.

Now fix $\epsilon > 0$ and let $W_n \in \mathcal{U}$, be such that $E \cup F \subseteq \bigcup_n W_n$ and

$$\sum_n \rho(W_n) \leq \mu^*(E \cup F) + \epsilon \leq \mu^*(E) + \mu^*(F) + \epsilon. \tag{$*$}$$

Note that $\{W_n \cap U\}_{n\in\mathbb{N}}$ covers E, $\{W_n \cap V\}_{n\in\mathbb{N}}$ covers F and $W_n \cap U, W_n \cap V \in \mathcal{U}$. Also, by finite additivity of ρ,

$$\rho(W_n \cap U) + \rho(W_n \cap V) = \rho(W_n \cap (U \cup V)) \leq \rho(W_n).$$

Thus

$$\mu^*(E) + \mu^*(F) \leq \sum_n \rho(W_n \cap U) + \sum_n \rho(W_n \cap V) \leq \sum_n \rho(W_n),$$

which, together with $(*)$, implies that $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$ since ϵ is arbitrary. □

Proposition 4.2. *Suppose there exist a countable base $\mathcal{U} \subseteq Pow(X)$ for X and a compact set $K \subseteq X$ such that the G-algebra generated by $\mathcal{U} \cup \{K\}$ admits a finitely additive G-invariant probability measure ρ with $\rho(K) > 0$. Then there exists a countably additive G-invariant Borel probability measure on X.*

Proof. Let K,$\mathcal{U}$ and ρ be as in the hypothesis. We may assume that $\mathcal{U}$ is closed under the G-action and finite unions/intersections. Let μ^* be the outer measure provided by Lemma 4.1 applied to $\mathcal{U}$, ρ. Thus μ^* is a metric outer measure on K and hence all Borel subsets of K are μ^*-measurable (see 13.2 in [Mun53]). This implies that all Borel subsets of $Y = [K]_G = \bigcup_{g\in G} gK$ are μ^*-measurable because μ^* is G-invariant. By Carathéodory's theorem, the restriction of μ^* to the Borel subsets of Y is a countably additive Borel measure on Y, and we extend it to a Borel measure μ on X by setting $\mu(Y^c) = 0$. Note that μ is G-invariant and $\mu(Y) \leq 1$.

It remains to show that μ is nontrivial, which we do by showing that $\mu(K) \geq \rho(K)$ and hence $\mu(K) > 0$. To this end, let $\{U_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ cover K. Since K is compact, there is a finite subcover $\{U_n\}_{n<N}$. Thus $U := \bigcup_{n<N} U_n \in \mathcal{U}$ and $K \subseteq U$. By finite additivity of ρ, we have

$$\sum_{n\in\mathbb{N}} \rho(U_n) \geq \sum_{n<N} \rho(U_n) \geq \rho(U) \geq \rho(K),$$

and hence, it follows from the definition of μ^* that $\mu^*(K) \geq \rho(K)$. Thus $\mu(K) = \mu^*(K) > 0$. □

Corollary 4.3. *Let X be a second countable Hausdorff topological G-space whose Borel structure is standard. For every compact set $K \subseteq X$ not in $\mathfrak{C}_4$, there is a G-invariant countably additive Borel probability measure μ on X with $\mu(K) > 0$.*

Proof. Fix any countable base $\mathcal{U}$ for X and let $\mathcal{B}$ be the Boolean G-algebra generated by $\mathcal{U} \cup \{K\}$. By Corollary 3.16, there exists a G-invariant finitely additive probability measure ρ on $\mathcal{B}$ such that $\rho(K) > 0$. Now apply 4.2. □

As a corollary, we derive the analogue of Nadkarni's theorem for $\mathfrak{C}_4$ in case of σ-compact spaces.

Corollary 4.4. *Let X be a Borel G-space that admits a σ-compact realization. $X \notin \mathfrak{C}_4$ if and only if there exists a G-invariant countably additive Borel probability measure on X.*

Proof. $\Leftarrow$: If $X \in \mathfrak{C}_4$, then it is compressible in the usual sense and hence does not admit a G-invariant Borel probability measure.
$\Rightarrow$: Suppose that X is a σ-compact topological G-space and $X \notin \mathfrak{C}_4$. Then, since X is σ-compact and $\mathfrak{C}_4$ is a σ-ideal, there is a compact set K not in $\mathfrak{C}_4$. Now apply 4.3. □

Remark. For a Borel G-space X, let $\mathcal{K}$ denote the collection of all subsets of invariant Borel sets that admit a σ-compact realization (when viewed as Borel G-spaces). Also, let $\mathfrak{C}$ denote the collection of all subsets of invariant compressible Borel sets. It is clear that $\mathcal{K}$ and $\mathfrak{C}$ are σ-ideals, and what 4.4 implies is that $\mathfrak{C} \cap \mathcal{K} \subseteq \mathfrak{C}_4$. The question of whether $\mathcal{K} = Pow(X)$ is just a rephrasing of §10.(B) of Introduction.

Theorem 4.5. *Let X be a Borel G-space that admits a σ-compact realization. If there is no G-invariant Borel probability measure on X, then X admits a Borel 32-generator.*

Proof. By 4.4, $X \in \mathfrak{C}_4$ and hence, X is 4-compressible. Thus, by Proposition 2.30, X admits a Borel 2^5-generator. □

Example 4.6. Let $LO \subseteq 2^{\mathbb{N}^2}$ denote the Polish space of all countable linear orderings and let G be the group of finite permutations of elements of $\mathbb{N}$. G is countable and acts continuously on LO in the natural way. Put $X = LO \setminus DLO$, where DLO denotes the set of all dense linear orderings without endpoints (copies of $\mathbb{Q}$). It is straightforward to see that DLO is a G_δ subset of LO and hence, X is F_σ. Therefore, X is in fact σ-compact since LO is compact being a closed subset of $2^{\mathbb{N}^2}$. Also note that X is G-invariant.

Let μ be the unique measure on LO defined by $\mu(V_{(F,<)}) = \frac{1}{n!}$, where $(F,<)$ is a finite linearly ordered subset of $\mathbb{N}$ of cardinality n and $V_{(F,<)}$ is the set of all linear orderings of $\mathbb{N}$ extending the order $<$ on F. As shown in [GW02], μ is the unique invariant measure for the action of G on LO and $\mu(X) = 0$. Thus there is no G-invariant Borel probability measure on X and hence, by the above theorem, X admits a Borel 32-generator.

5. Finitely traveling sets

Let X be a Borel G-space.

Definition 5.1. *Let $A, B \in \mathfrak{B}(X)$ be equidecomposable, i.e. there are $N \leq \infty$, $\{g_n\}_{n<N} \subseteq G$ and Borel partitions $\{A_n\}_{n<N}$ and $\{B_n\}_{n<N}$ of A and B, respectively, such that $g_n A_n = B_n$ for all $n < N$. A, B are said to be*

- *locally finitely equidecomposable (denote by $A \sim_{lfin} B$), if $\{A_n\}_{n<N}, \{B_n\}_{n<N}, \{g_n\}_{n<N}$ can be taken so that for every $x \in A$, $A_n \cap [x]_G = \emptyset$ for all but finitely many $n < N$;*
- *finitely equidecomposable (denote by $A \sim_{fin} B$), if N can be taken to be finite.*

The notation $\prec_{\text{fin}}$, $\prec_{\text{lfin}}$ and the notions of finite and locally finite compressibility are defined analogous to Definitions 1.13 and 1.15.

Definition 5.2. *A Borel set $A \subseteq X$ is called (locally) finitely traveling if there exists pairwise disjoint Borel sets $\{A_n\}_{n\in\mathbb{N}}$ such that $A_0 = A$ and $A \sim_{fin} A_n$ ($A \sim_{lfin} A_n$), $\forall n \in \mathbb{N}$.*

Proposition 5.3. *If X is (locally) finitely compressible then X admits a (locally) finitely traveling Borel complete section.*

Proof. We prove for finitely compressible X, but note that everything below is also locally valid (i.e. restricted to every orbit) for a locally compressible X.

Run the proof of the first part of Lemma 2.21 noting that a witnessing map $\gamma : X \to G$ of finite compressibility of X has finite image and hence the image of each δ_n (in the notation of the proof) is finite, which implies that the obtained traveling set A is actually finitely traveling. □

Proposition 5.4. *If X admits a locally finitely traveling Borel complete section, then $X \in \mathfrak{C}_4$.*

Proof. Let A be a locally finitely traveling Borel complete section and let $\{A_n\}_{n\in\mathbb{N}}$ be as in Definition 5.2. Let $\mathcal{I}_n = \{C^n_k\}_{k\in\mathbb{N}}$, $\mathcal{J}_n = \{D^n_k\}_{k\in\mathbb{N}}$ be Borel partitions of A and A_n, respectively, that together with $\{g^n_k\}_{k\in\mathbb{N}} \subseteq G$ witness $A \sim_{\text{lfin}} A_n$ (as in Definition 5.1). Let $\mathcal{B}$ denote the Boolean G-algebra generated by $\{X\} \cup \bigcup_{n\in\mathbb{N}}(\mathcal{I}_n \cup \mathcal{J}_n \cup \{A_n\})$.

Now assume for contradiction that $X \notin \mathfrak{C}_4$ and hence, $A \notin \mathfrak{C}_4$. Thus, applying Corollary 3.16 to A and $\mathcal{B}$, we get a G-invariant finitely additive probability measure μ on $\mathcal{B}$ with $\mu(A) > 0$. Moreover, there is $x \in A$ such that $\forall B \in \mathcal{B}$ with $B \cap [x]_G = \emptyset$, $\mu(B) = 0$.

Claim. *$\mu(A_n) = \mu(A)$, for all $n \in \mathbb{N}$.*

Proof of Claim. For each n, let $\{C^n_{k_i}\}_{i<K_n}$ be the list of those C^n_k such that $C^n_k \cap [x]_G \neq \emptyset$ ($K_n < \infty$ by the definition of locally finitely traveling). Set $B = A \setminus (\bigcup_{i<K_n} C^n_{k_i})$ and note that by finite additivity of μ,

$$\mu(A) = \mu(B) + \sum_{i<K_n} \mu(C^n_{k_i}).$$

Similarly, set $B' = A_n \setminus (\bigcup_{i<K_n} D^n_{k_i})$ and hence

$$\mu(A_n) = \mu(B') + \sum_{i<K_n} \mu(D^n_{k_i}).$$

But $B \cap [x]_G = \emptyset$ and $B' \cap [x]_G = \emptyset$, and thus $\mu(B) = \mu(B') = 0$. Also, since $g^n_{k_i} C^n_{k_i} = D^n_{k_i}$ and μ is G-invariant, $\mu(C^n_{k_i}) = \mu(D^n_{k_i})$. Therefore

$$\mu(A) = \sum_{i<K_n} \mu(C^n_{k_i}) = \sum_{i<K_n} \mu(D^n_{k_i}) = \mu(A_n).$$

$\dashv$

This claim contradicts μ being a probability measure since for large enough N, $\mu(\bigcup_{n<N} A_n) = N\mu(A) > 1$, contradicting $\mu(X) = 1$. □

This, together with 2.30, implies the following.

Corollary 5.5. *Let X be a Borel G-space. If X admits a locally finitely traveling Borel complete section, then there is a Borel 32-generator.*

6. Separating smooth-many invariant sets

Assume throughout that X is a Borel G-space.

Lemma 6.1. *If X is aperiodic then it admits a countably infinite partition into Borel complete sections.*

Proof. The following argument is also given in the proof of Theorem 13.1 in [KM04]. By the marker lemma (see 6.7 in [KM04]), there exists a vanishing sequence $\{B_n\}_{n\in\mathbb{N}}$ of decreasing Borel complete sections, i.e. $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$. For each $n \in \mathbb{N}$, define $k_n : X \to \mathbb{N}$ recursively as follows:

$$\begin{cases} k_0(x) &= 0 \\ k_{n+1}(x) &= min\{k \in \mathbb{N} : B_{k_n(x)} \cap [x]_G \not\subseteq B_k\} \end{cases},$$

and define $A_n \subseteq X$ by

$$x \in A_n \Leftrightarrow x \in A_{k_n(x)} \setminus A_{k_{n+1}(x)}.$$

It is straightforward to check that A_n are pairwise disjoint Borel complete sections. □

For $A \in \mathfrak{B}(X)$, if $\mathcal{I} = <A>$ then we use the notation F_A and f_A instead of $F_\mathcal{I}$ and $f_\mathcal{I}$, respectively.

We now work towards strengthening the above lemma to yield a countably infinite partition into F_A-invariant Borel complete sections.

Definition 6.2 (Aperiodic separation)**.** *For Borel sets $A, Y \subseteq X$, we say that A aperiodically separates Y if $f_A([Y]_G)$ is aperiodic (as an invariant subset of the shift 2^G). If such A exists, we say that Y is aperiodically separable.*

Proposition 6.3. *For $A \in \mathfrak{B}(X)$, if A aperiodically separates X, then X admits a countably infinite partition into Borel F_A-invariant complete sections.*

Proof. Let $Y = \{y \in 2^G : |[y]_G| = \infty\}$ and hence $f_A(X)$ is a G-invariant subset of Y. By Lemma 6.1 applied to Y, there is a partition $\{B_n\}_{n\in\mathbb{N}}$ of Y into Borel complete sections. Thus $A_n = f_{\mathcal{I}}^{-1}(B_n)$ is a Borel F_A-invariant complete section for X and $\{A_n\}_{n\in\mathbb{N}}$ is a partition of X. □

Let $\mathfrak{A}$ denote the collection of all subsets of aperiodically separable Borel sets.

Lemma 6.4. *$\mathfrak{A}$ is a σ-ideal.*

Proof. We only have to show that if Y_n are aperiodically separable Borel sets, then $Y = \bigcup_{n\in\mathbb{N}} Y_n \in \mathfrak{A}$. Let A_n be a Borel set aperiodically separating Y_n. Since A_n also aperiodically separates $[Y_n]_G$ (by definition), we can assume that Y_n is G-invariant. Furthermore, by taking $Y_n' = Y_n \setminus \bigcup_{k<n} Y_k$, we can assume that Y_n are pairwise disjoint. Now letting $A = \bigcup_{n\in\mathbb{N}}(A_n \cap Y_n)$, it is easy to check that A aperiodically separates Y. □

Let $\mathfrak{S}$ denote the collection of all subsets of smooth sets. By a similar argument as the one above, $\mathfrak{S}$ is a σ-ideal.

Lemma 6.5. *If X is aperiodic, then $\mathfrak{S} \subseteq \mathfrak{A}$.*

Proof. Let $S \in \mathfrak{S}$ and hence there is a Borel transversal T for $[S]_G$. Fix $x \in S$ and let $y \neq z \in [x]_G$. Since T is a transversal, there is $g \in G$ such that $gy \in T$, and hence $gz \notin T$. Thus $f_T(y) \neq f_T(z)$, and so $f_T([x]_G)$ is infinite. Therefore T aperiodically separates $[S]_G$. □

For the rest of the section, fix an enumeration $G = \{g_n\}_{n\in\mathbb{N}}$ and let F_A^n be following equivalence relation:

$$yF_A^n z \Leftrightarrow \forall k < n(g_k y \in A \leftrightarrow g_k z \in A).$$

Note that F_A^n has no more than 2^n equivalence classes and that yF_Az if and only if $\forall n(yF_A^n z)$.

Lemma 6.6. *For $A, Y \in \mathfrak{B}(X)$, A aperiodically separates Y if and only if* $(\forall x \in Y)(\forall n)(\exists y, z \in Y^{[x]_G})[yF_A^n z \wedge \neg(yF_A z)]$.

Proof. $\Rightarrow$: Assume that for all $x \in Y$, $f_A([x]_G)$ is infinite and thus $F_A|_{[x]_G}$ has infinitely many equivalence classes. Fix $n \in \mathbb{N}$ and recall that F_A^n has only finitely many equivalence classes. Thus, by the Pigeon Hole Principle, there are $y, z \in Y^{[x]_G}$ such that $yF_A^n z$ yet $\neg(yF_Az)$.
$\Leftarrow$: Assume for contradiction that $f_A(Y^{[x]_G})$ is finite for some $x \in Y$. Then it follows that $F_A = F_A^n$, for some n, and hence for any $y, z \in Y^{[x]_G}$, $yF_A^n z$ implies yF_Az, contradicting the hypothesis. □

Theorem 6.7. *If X is an aperiodic Borel G-space, then $X \in \mathfrak{A}$.*

Proof. By Lemma 6.1, there is a partition $\{A_n\}_{n\in\mathbb{N}}$ of X into Borel complete sections. We will inductively construct Borel sets $B_n \subseteq C_n$, where C_n should be thought of as the set of points *colored* (black or white) at the n^{th} step, and B_n as the set of points colored *black* (thus $C_n \setminus B_n$ is colored *white*).

Define a function $\# : X \to \mathbb{N}$ by $x \mapsto m$, where m is such that $x \in A_m$. Fix a countable family $\{U_n\}_{n\in\mathbb{N}}$ of sets generating the Borel σ-algebra of X.

Assuming that for all $k < n$, C_k, B_k are defined, let $\bar{C}_n = \bigcup_{k<n} C_k$ and $\bar{B}_n = \bigcup_{k<n} B_k$. Put $P_n = \{x \in A_0 : \forall k < n(g_k x \in \bar{C}_n) \wedge g_n x \notin \bar{C}_n\}$ and set $F_n = F^n_{\bar{B}_n}\downharpoonright_{P_n}$, that is for all $x, y \in P_n$,

$$yF_n z \Leftrightarrow \forall k < n(g_k y \in \bar{B}_n \leftrightarrow g_k z \in \bar{B}_n).$$

Now put $C'_n = \{x \in P_n : \#(g_n x) = \min \#((g_n P_n)^{[x]_G})\}$, $C''_n = \{x \in C'_n : \exists y, z \in (C'_n)^{[x]_G}(y \neq z \wedge yF_n z)\}$ and $C_n = g_n C''_n$. Note that it follows from the definition of P_n that C_n is disjoint from $\bar{C}_n$.

Now in order to define B_n, first define a function $\bar{n} : X \to \mathbb{N}$ by

$$x \mapsto \text{ the smallest } m \text{ such that there are } y, z \in C''_n \cap [x]_G \text{ with } yF_n z, y \in U_m \text{ and } z \notin U_m.$$

Note that $\bar{n}$ is Borel and G-invariant. Lastly, let $B'_n = \{x \in C''_n : x \in U_{\bar{n}(x)}\}$ and $B_n = g_n B'_n$. Clearly $B_n \subseteq C_n$. Now let $B = \bigcup_{n\in\mathbb{N}} B_n$ and $D = \left[\bigcup_{n\in\mathbb{N}}(C'_n \setminus C''_n)\right]_G$. We show that B aperiodically separates $Y := X \setminus D$ and $D \in \mathfrak{S}$. Since $\mathfrak{S} \subseteq \mathfrak{A}$ and $\mathfrak{A}$ is an ideal, this will imply that $X \in \mathfrak{A}$.

Claim 1. *$D \in \mathfrak{S}$.*

Proof of Claim. Since $\mathfrak{S}$ is a σ-ideal, it is enough to show that for each n, $[C'_n \setminus C''_n]_G \in \mathfrak{S}$, so fix $n \in \mathbb{N}$. Clearly $(C'_n \setminus C''_n)^{[x]_G}$ is finite, for all $x \in X$, since there can be at most 2^n pairwise F_n-nonequivalent points. Thus, fixing some Borel linear ordering of X and taking the smallest element from $(C'_n \setminus C''_n)^{[x]_G}$ for each $x \in C'_n \setminus C''_n$, we can define a Borel transversal for $[C'_n \setminus C''_n]_G$. ⊣

By Lemma 6.6, to show that B aperiodically separates Y, it is enough to show that $(\forall x \in Y)(\forall n)(\exists y, z \in [x]_G)[yF^n_B z \wedge \neg(yF_B z)]$. Fix $x \in Y$.

Claim 2. *$(\exists^\infty n)(C''_n)^{[x]_G} \neq \emptyset$.*

Proof of Claim. Assume for contradiction that $(\forall^\infty n)(C''_n)^{[x]_G} = \emptyset$. Since $x \notin D$, it follows that $(\forall^\infty n)P_n^{[x]_G} = \emptyset$. Since A_0 is a complete section and $\bar{C}_0 = \emptyset$, $P_0^{[x]_G} \neq \emptyset$. Let N be the largest number such that $P_N^{[x]_G} \neq \emptyset$. Thus for all $n > N$, $C_n^{[x]_G} = \emptyset$ and hence for all $n > N$, $\bar{C}_n^{[x]_G} = \bar{C}_{N+1}^{[x]_G}$. Because $C_N^{[x]_G} \neq \emptyset$, there is $y \in A_0^{[x]_G}$ such that $\forall k \leq N(g_k y \in \bar{C}_{N+1})$; but because $P_{N+1}^{[x]_G} = \emptyset$, $g_{N+1} y$ must also fall into $\bar{C}_{N+1}$. By induction on $n > N$, we get that for all $n > N$, $g_n y \in \bar{C}_n$ and thus $g_n y \in \bar{C}_{N+1}$.

On the other hand, it follows from the definition of C'_n that for each n, $(C'_n)^{[x]_G}$ intersects exactly one of A_k. Thus $\bar{C}_{N+1}^{[x]_G}$ intersects at most $N+1$ of A_k and hence there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $\bar{C}_{N+1}^{[x]_G} \cap A_k = \emptyset$. Since $\exists^\infty n(g_n y \in \bigcup_{k\geq K} A_k)$, $\exists^\infty n(g_n y \notin \bar{C}_{N+1})$, a contradiction. ⊣

Now it remains to show that for all $n \in \mathbb{N}$, $(C''_n)^{[x]_G} \neq \emptyset$ implies that $\exists y, z \in [x]_G$ such that $yF^n_B z$ but $\neg(yF_B z)$. To this end, fix $n \in \mathbb{N}$ and assume $(C''_n)^{[x]_G} \neq \emptyset$. Thus there are $y, z \in (C''_n)^{[x]_G}$ such that $yF_n z$, $y \in U_{\bar{n}(x)}$ and $z \notin U_{\bar{n}(x)}$; hence, $g_n y \in B_n$ and $g_n z \notin B_n$, by the definition of B_n. Since C_k are pairwise disjoint, $B_n \subseteq C_n$ and $g_n y, g_n z \in C_n$, it follows that $g_n y \in B$ and $g_n z \notin B$, and therefore $\neg(yF_B z)$. Finally, note that $F_n = F^n_B\downharpoonright_{P_n}$ and hence $yF^n_B z$. □

Corollary 6.8. *Suppose all of the nontrivial subgroups of G have finite index (e.g. $G = \mathbb{Z}$), and let X be an aperiodic Borel G-space. Then there exists $A \in \mathfrak{B}(X)$ such that G <A> separates points in each orbit, i.e. $f_A\downharpoonright_{[x]_G}$ is one-to-one, for all $x \in X$.*

Proof. Let A be a Borel set aperiodically separating X (exists by Theorem 6.7) and put $Y = f_A(X)$. Then $Y \subseteq 2^G$ is aperiodic and hence the action of G on Y is free since the stabilizer subgroup of every element must have infinite index and thus is trivial. But this implies that for all $y \in Y$, $f_A^{-1}(y)$ intersects every orbit in X at no more than one point, and hence f_A is one-to-one on every orbit. □

From 6.3 and 6.7 we immediately get the following strengthening of Lemma 6.1.

Corollary 6.9. *If X is aperiodic, then for some $A \in \mathfrak{B}(X)$, X admits a countably infinite partition into Borel F_A-invariant complete sections.*

Theorem 6.10. *Let X be an aperiodic G-space and let E be a smooth equivalence relation on X with $E_G \subseteq E$. There exists a partition $\mathcal{P}$ of X into 4 Borel sets such that $G\mathcal{P}$ separates any two E-nonequivalent points in X, i.e. $\forall x, y \in X(\neg(xEy) \to f_{\mathcal{P}}(x) \neq f_{\mathcal{P}}(y))$.*

Proof. By Corollary 6.9, there is $A \in \mathfrak{B}(X)$ and a Borel partition $\{A_n\}_{n\in\mathbb{N}}$ of X into F_A-invariant complete sections. For each $n \in \mathbb{N}$, define a function $\bar{n} : X \to \mathbb{N}$ by

$$x \mapsto \text{the smallest } m \text{ such that } \exists x' \in A_0^{[x]_G} \text{ with } g_m x' \in A_n.$$

Clearly $\bar{n}$ is Borel, and because all of A_k are F_A-invariant, $\bar{n}$ is also F_A-invariant, i.e. for all $x, y \in X$, $xF_Ay \to \bar{n}(x) = \bar{n}(y)$. Also, $\bar{n}$ is G-invariant by definition.

Put $A'_n = \{x \in A_0 : g_{\bar{n}(x)}x \in A_n\}$ and note that A'_n is F_A-invariant Borel since so are $\bar{n}$, A_0 and A_n. Moreover, A'_n is clearly a complete section. Define $\gamma_n : A'_n \to A_n$ by $x \mapsto g_{\bar{n}(x)}x$. Clearly, γ_n is Borel and one-to-one.

Since E is smooth, there is a Borel $h : X \to \mathbb{R}$ such that for all $x, y \in X$, $xEy \leftrightarrow h(x) = h(y)$. Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable family of subsets of $\mathbb{R}$ generating the Borel σ-algebra of $\mathbb{R}$ and put $U_n = h^{-1}(V_n)$. Because each equivalence class of E is G-invariant, so is h and hence so is U_n.

Now let $B_n = \gamma_n(A'_n \cap U_n)$ and note that B_n is Borel being a one-to-one Borel image of a Borel set. It follows from the definition of γ_n that $B_n \subseteq A_n$. Put $B = \bigcup_{n\in\mathbb{N}} B_n$ and $\mathcal{P} = <A, B>$; in particular, $|\mathcal{P}| \leq 4$. We show that $\mathcal{P}$ is what we want. To this end, fix $x, y \in X$ with $\neg(xEy)$. If $\neg(xF_Ay)$, then $G<A>$ (and hence $G\mathcal{P}$) separates x and y.

Thus assume that xF_Ay. Since $h(x) \neq h(y)$, there is n such that $h(x) \in V_n$ and $h(y) \notin V_n$. Hence, by invariance of U_n, $gx \in U_n \wedge gy \notin U_n$, for all $g \in G$. Because A'_n is a complete section, there is $g \in G$ such that $gx \in A'_n$ and hence $gy \in A'_n$ since A'_n is F_A-invariant. Let $m = \bar{n}(gx)$ $(= \bar{n}(gy))$. Then $g_m gx \in B_n$ while $g_m gy \notin B_n$ although $g_m gy \in \gamma_n(A'_n) \subseteq A_n$. Thus $g_m gx \in B$ but $g_m gy \notin B$ and therefore $G\mathcal{P}$ separates x and y. □

7. Potential dichotomy theorems

In this section we prove dichotomy theorems assuming Weiss's question has a positive answer for $G = \mathbb{Z}$. In the proofs we use the Ergodic Decomposition Theorem (see [Far62], [Var63]) and a Borel/uniform version of Krieger's finite generator theorem, so we first state both of the theorems and sketch the proof of the latter.

For a Borel G-space X, let $\mathcal{M}_G(X)$ denote the set of G-invariant Borel probability measures on X and let $\mathcal{E}_G(X)$ denote the set of ergodic ones among those. Clearly both are Borel subsets of $P(X)$ (the standard Borel space of Borel probability measures on X) and thus are themselves standard Borel spaces.

Ergodic Decomposition Theorem 7.1 (Farrell, Varadarajan)**.** *Let X be a Borel G-space. If $\mathcal{M}_G(X) \neq \emptyset$ (and hence $\mathcal{E}_G(X) \neq \emptyset$), then there is a Borel surjection $x \mapsto e_x$ from X onto $\mathcal{E}_G(X)$ such that:*

(i) $xE_G y \Rightarrow e_x = e_y$;

(ii) For each $e \in \mathcal{E}_G(X)$, if $X_e = \{x \in X : e_x = e\}$ (hence X_e is invariant Borel), then $e(X_e) = 1$ and $e\lfloor_{X_e}$ is the unique ergodic invariant Borel probability measure on X_e;

(iii) For each $\mu \in \mathcal{M}_G(X)$ and $A \in \mathfrak{B}(X)$, we have $\mu(A) = \int e_x(A) d\mu(x)$.

For the rest of the section, let X be a Borel $\mathbb{Z}$-space.

For $e \in \mathcal{E}_{\mathbb{Z}}(X)$, if we let h_e denote the entropy of $(X, \mathbb{Z}, e)$, then the map $e \mapsto h_e$ is Borel. Indeed, if $\{\mathcal{P}_k\}_{k \in \mathbb{N}}$ is a refining sequence of partitions of X that generates the Borel σ-algebra of X, then by 4.1.2 of [Dow11], $h_e = \lim_{k \to \infty} h_e(\mathcal{P}_k, \mathbb{Z})$, where $h_e(\mathcal{P}_k, \mathbb{Z})$ denotes the entropy of $\mathcal{P}_k$. By 17.21 of [Kec95], the function $e \mapsto h_e(\mathcal{P}_k)$ is Borel and thus so is the map $e \mapsto h_e$.

For all $e \in \mathcal{E}_{\mathbb{Z}}(X)$ with $h_e < \infty$, let N_e be the smallest integer such that $\log N_e > h_e$. The map $e \mapsto N_e$ is Borel because so is $e \mapsto h_e$.

Krieger's Finite Generator Theorem 7.2 (Uniform version)**.** *Let X be a Borel $\mathbb{Z}$-space. Suppose $\mathcal{M}_{\mathbb{Z}}(X) \neq \emptyset$ and let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Assume also that all measures in $\mathcal{E}_{\mathbb{Z}}(X)$ have finite entropy and let $e \mapsto N_e$ be the map defined above. Then there is a partition $\{A_n\}_{n \leq \infty}$ of X into Borel sets such that*

(i) A_∞ is invariant and does not admit an invariant Borel probability measure;

(ii) For each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $\{A_n \cap X_e\}_{n < N_e}$ is a generator for $X_e \setminus A_\infty$, where $X_e = \rho^{-1}(e)$.

Sketch of Proof. Note that it is enough to find a Borel invariant set $X' \subseteq X$ and a Borel $\mathbb{Z}$-map $\phi : X' \to \mathbb{N}^{\mathbb{Z}}$, such that for each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, we have

(I) $e(X \setminus X') = 0$;

(II) $\phi\lfloor_{X_e \cap X'}$ is one-to-one and $\phi(X_e \cap X') \subseteq (N_e)^{\mathbb{Z}}$, where $(N_e)^{\mathbb{Z}}$ is naturally viewed as a subset of $\mathbb{N}^{\mathbb{Z}}$.

Indeed, assume we had such X' and ϕ, and let $A_\infty = X \setminus X'$ and $A_n = \phi^{-1}(V_n)$ for all $n \in \mathbb{N}$, where $V_n = \{y \in \mathbb{N}^{\mathbb{Z}} : y(0) = n\}$. Then it is clear that $\{A_n\}_{n \in \mathbb{N}}$ satisfies (ii). Also, (I) and part (ii) of the Ergodic Decomposition Theorem imply that (i) holds for A_∞.

To construct such a ϕ, we use the proof of Krieger's theorem presented in [Dow11], Theorem 4.2.3, and we refer to it as Downarowicz's proof. For each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, the proof constructs a Borel $\mathbb{Z}$-embedding $\phi_e : X' \to N_e^{\mathbb{Z}}$ on an e-measure 1 set X'. We claim that this construction is uniform in e in a Borel way and hence would yield X' and ϕ as above.

Our claim can be verified by inspection of Downarowicz's proof. The proof uses the existence of sets with certain properties and one has to check that such sets exist with the properties satisfied for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$ at once. For example, the set C used in the proof of Lemma 4.2.5 in [Dow11] can be chosen so that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $C \cap X_e$ has the required properties for e (using the Shannon-McMillan-Brieman theorem). Another example is the set B used in the proof of the same lemma, which is provided by Rohlin's lemma. By inspection of the proof of Rohlin's lemma (see 2.1 in [Gla03]), one can verify that we can get a Borel B such that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $B \cap X_e$ has the required properties for e. The sets in these two examples are the only kind of sets whose existence is used in the whole proof; the rest of the proof constructs the required ϕ "by hand". □

Theorem 7.3 (Dichotomy I)**.** *Suppose the answer to Question 1.6 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant ergodic Borel probability measure with infinite entropy;
(2) there exists a partition $\{Y_n\}_{n\in\mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator.

Proof. We first show that the conditions above are mutually exclusive. Indeed, assume there exist an invariant ergodic Borel probability measure e with infinite entropy and a partition $\{Y_n\}_{n\in\mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator. By ergodicity, e would have to be supported on one of the Y_n. But Y_n has a finite generator and hence the dynamical system $(Y_n, \mathbb{Z}, e)$ has finite entropy by the Kolmogorov-Sinai theorem (see 1.4). Thus so does $(X, \mathbb{Z}, e)$ since these two systems are isomorphic (modulo e-NULL), contradicting the assumption on e.

Now we prove that at least one of the conditions holds. Assume that there is no invariant ergodic measure with infinite entropy. Now, if there was no invariant Borel probability measure at all, then, since the answer to Question 1.6 is assumed to be positive, X would admit a finite generator, and we would be done. So assume that $\mathcal{M}_{\mathbb{Z}}(X) \neq \emptyset$ and let $\{A_n\}_{n\leq\infty}$ be as in Theorem 7.2. Furthermore, let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Set $X' = X \setminus A_\infty$, $Y_\infty = A_\infty$, and for all $n \in \mathbb{N}$,

$$Y_n = \{x \in X' : N_{e_x} = n\},$$

where the map $e \mapsto N_e$ is as above. Note that the sets Y_n are invariant since ρ is invariant, so $\{Y_n\}_{n\leq\infty}$ is a countable partition of X into invariant Borel sets. Since Y_∞ does not admit an invariant Borel probability measure, by our assumption, it has a finite generator.

Let E be the equivalence relation on X' defined by ρ, i.e. $\forall x, y \in X'$,

$$xEy \Leftrightarrow \rho(x) = \rho(y).$$

By definition, E is a smooth Borel equivalence relation with $E \supseteq E_{\mathbb{Z}}$ since ρ respects the $\mathbb{Z}$-action. Thus, by Theorem 6.10, there exists a partition $\mathcal{P}$ of X' into 4 Borel sets such that $\mathbb{Z}\mathcal{P}$ separates any two points in different E-classes.

Now fix $n \in \mathbb{N}$ and we will show that $\mathcal{I} = \mathcal{P} \vee \{A_i\}_{i<n}$ is a generator for Y_n. Indeed, take distinct $x, y \in Y_n$. If x and y are in different E-classes, then $\mathbb{Z}\mathcal{P}$ separates them and hence so does $\mathbb{Z}\mathcal{I}$. Thus we can assume that xEy. Then $e := \rho(x) = \rho(y)$, i.e. $x, y \in X_e = \rho^{-1}(e)$. By the choice of $\{A_i\}_{i\in\mathbb{N}}$, $\{A_n \cap X_e\}_{n<N_e}$ is a generator for X_e and hence $\mathbb{Z}\{A_i\}_{i<N_e}$ separates x and y. But $n = N_e$ by the definition of Y_n, so $\mathbb{Z}\mathcal{I}$ separates x and y. □

Proposition 7.4. *Let X be a Borel $\mathbb{Z}$-space. If X admits invariant ergodic probability measures of arbitrarily large entropy, then it admits an invariant probability measure of infinite entropy.*

Proof. For each $n \geq 1$, let μ_n be an invariant ergodic probability measure of entropy $h_{\mu_n} > n2^n$ such that $\mu_n \neq \mu_m$ for $n \neq m$, and put

$$\mu = \sum_{n\geq 1} \frac{1}{2^n}\mu_n.$$

It is clear that μ is an invariant probability measure, and we show that its entropy h_μ is infinite. Fix $n \geq 1$. Let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem and put $X_n = \rho^{-1}(\mu_n)$. It is clear that $\mu_m(X_n) = 1$ if $m = n$ and 0 otherwise.

For any finite Borel partition $\mathcal{P} = \{A_i\}_{i=1}^k$ of X_n, put $A_0 = X \setminus X_n$ and $\bar{\mathcal{P}} = \mathcal{P} \cup \{A_0\}$. Let T be the Borel automorphism of X corresponding to the action of $1_{\mathbb{Z}}$, and let $h_\nu(\mathcal{I})$ and $h_\nu(\mathcal{I}, T)$ denote, respectively, the static and dynamic entropies of a finite Borel partition $\mathcal{I}$

of X with respect to an invariant probability measure ν. Then, with the convention that $\log(0) \cdot 0 = 0$, we have

$$h_\mu(\bar{\mathcal{P}}) = -\sum_{i=0}^{k} \log(\mu(A_i))\mu(A_i) \geq -\sum_{i=1}^{k} \log(\mu(A_i))\mu(A_i) = -\sum_{i=1}^{k} \log(\frac{1}{2^n}\mu_n(A_i))\frac{1}{2^n}\mu_n(A_i)$$
$$\geq -\frac{1}{2^n}\sum_{i=1}^{k} \log(\mu_n(A_i))\mu_n(A_i) = \frac{1}{2^n} h_{\mu_n}(\bar{\mathcal{P}}).$$

Since $\mathcal{P}$ is arbitrary and X_n is invariant, it follows that

$$h_\mu(\bar{\mathcal{P}}, T) = \lim_{m\to\infty} \frac{1}{m} h_\mu(\bigvee_{j<m} T^j \bar{\mathcal{P}}) \geq \frac{1}{2^n} \lim_{m\to\infty} \frac{1}{m} h_{\mu_n}(\bigvee_{j<m} T^j \bar{\mathcal{P}}) = \frac{1}{2^n} h_{\mu_n}(\bar{\mathcal{P}}, T).$$

Now for any finite Borel partition $\mathcal{I}$ of X, it is clear that $h_{\mu_n}(\mathcal{I}) = h_{\mu_n}(\bar{\mathcal{P}})$ (and hence $h_{\mu_n}(\mathcal{I}, T) = h_{\mu_n}(\bar{\mathcal{P}}, T)$), for some $\mathcal{P}$ as above. This implies that

$$h_\mu \geq \sup_{\mathcal{P}} h_\mu(\bar{\mathcal{P}}, T) \geq \frac{1}{2^n} \sup_{\mathcal{P}} h_{\mu_n}(\bar{\mathcal{P}}, T) = \frac{1}{2^n} \sup_{\mathcal{I}} h_{\mu_n}(\mathcal{I}, T) = \frac{1}{2^n} h_{\mu_n} > n,$$

where $\mathcal{P}$ and $\mathcal{I}$ range over finite Borel partitions of X_n and X, respectively. Thus $h_\mu = \infty$. □

Theorem 7.5 (Dichotomy II). *Suppose the answer to Question 1.6 is positive and let X be an aperiodic Borel $\mathbb{Z}$-space. Then exactly one of the following holds:*

(1) there exists an invariant Borel probability measure with infinite entropy;
(2) X admits a finite generator.

Proof. The Kolmogorov-Sinai theorem implies that the conditions are mutually exclusive, and we prove that at least one of them holds. Assume that there is no invariant measure with infinite entropy. If there was no invariant Borel probability measure at all, then, by our assumption, X would admit a finite generator. So assume that $\mathcal{M}_{\mathbb{Z}}(X) \neq \emptyset$ and let $\{A_n\}_{n\leq\infty}$ be as in Theorem 7.2. Furthermore, let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Set $X' = X \setminus A_\infty$ and $X_e = \rho^{-1}(e)$, for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$.

By our assumption, A_∞ admits a finite generator $\mathcal{P}$. Also, by 7.4, there is $N \geq 1$ such that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $N_e \leq N$ and hence $\mathcal{Q} := \{A_n\}_{n<N}$ is a finite generator for X_e; in particular, $\mathcal{Q}$ is a partition of X'. Let E be the following equivalence relation on X:

$$xEy \Leftrightarrow (x, y \in A_\infty) \vee (x, y \in X' \wedge \rho(x) = \rho(y)).$$

By definition, E is a smooth equivalence relation with $E \supseteq E_{\mathbb{Z}}$ since ρ respects the $\mathbb{Z}$-action and A_∞ is $\mathbb{Z}$-invariant. Thus, by Theorem 6.10, there exists a partition $\mathcal{J}$ of X into 4 Borel sets such that $\mathbb{Z}\mathcal{J}$ separates any two points in different E-classes.

We now show that $\mathcal{I} :=<\mathcal{J} \cup \mathcal{P} \cup \mathcal{Q}>$ is a generator. Indeed, fix distinct $x, y \in X$. If x and y are in different E-classes, then $\mathbb{Z}\mathcal{J}$ separates them. So we can assume that xEy. If $x, y \in A_\infty$, then $\mathbb{Z}\mathcal{P}$ separates x and y. Finally, if $x, y \in X'$, then $x, y \in X_e$, where $e = \rho(x)$ $(= \rho(y))$, and hence $\mathbb{Z}\mathcal{Q}$ separates x and y. □

Remark. It is likely that the above dichotomies are also true for any amenable group using a uniform version of Krieger's theorem for amenable groups, cf. [DP02], but I have not checked the details.

8. Finite generators on comeager sets

Throughout this section let X be an aperiodic Polish G-space. We use the notation $\forall^*$ to mean "for comeager many x".

The following lemma proves the conclusion of Lemma 6.8 for any group on a comeager set. Below, we use this lemma only to conclude that there is an aperiodically separable comeager set, while we already know from 6.7 that X itself is aperiodically separable. However, the proof of the latter is more involved, so we present this lemma to keep this section essentially self-contained.

Lemma 8.1. *There exists $A \in \mathfrak{B}(X)$ such that $G <A>$ separates points in each orbit of a comeager G-invariant set D, i.e. $f_A|_{[x]_G}$ is one-to-one, for all $x \in D$.*

Proof. Fix a countable basis $\{U_n\}_{n\in\mathbb{N}}$ for X with $U_0 = \emptyset$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a partition of X provided by Lemma 6.1. For each $\alpha \in \mathcal{N}$ (the Baire space), define

$$B_\alpha = \bigcup_{n\in\mathbb{N}} (A_n \cap U_{\alpha(n)}).$$

Claim. $\forall^*\alpha \in \mathcal{N}\forall^* z \in X \forall x, y \in [z]_G(x \neq y \Rightarrow \exists g \in G(gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha))$.

Proof of Claim. By Kuratowski-Ulam, it is enough to show the statement with places of quantifiers $\forall^*\alpha \in \mathcal{N}$ and $\forall^* z \in X$ switched. Also, since orbits are countable and countable intersection of comeager sets is comeager, we can also switch the places of quantifiers $\forall^*\alpha \in \mathcal{N}$ and $\forall x, y \in [z]_G$. Thus we fix $z \in X$ and $x, y \in [z]_G$ with $x \neq y$ and show that $C = \{\alpha \in \mathcal{N} : \exists g \in G \ (gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha)\}$ is dense open.

To see that C is open, take $\alpha \in C$ and let $g \in G$ be such that $gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha$. Let $n, m \in \mathbb{N}$ be such that $gx \in A_n$ and $gy \in A_m$. Then for all $\beta \in \mathcal{N}$ with $\beta(n) = \alpha(n)$ and $\beta(m) = \alpha(m)$, we have $gx \in B_\beta \not\Leftrightarrow gy \in B_\beta$. But the set of such β is open in $\mathcal{N}$ and contained in C.

For the density of C, let $s \in \mathbb{N}^{<\mathbb{N}}$ and set $n = |s|$. Since A_n is a complete section, $\exists g \in G$ with $gx \in A_n$. Let $m \in \mathbb{N}$ be such that $gy \in A_m$. Take any $t \in \mathbb{N}^{\max\{n,m\}+1}$ with $t \sqsupseteq s$ satisfying the following condition:
Case 1: $n > m$. If $gy \in U_{s(m)}$ then set $t(n) = 0$. If $gy \notin U_{s(m)}$, then let k be such that $gx \in U_k$ and set $t(n) = k$.
Case 2: $n \leq m$. Let k be such that $gx \in U_k$ but $gy \notin U_k$ and set $t(n) = t(m) = k$.

Now it is easy to check that in any case $gx \in B_\alpha \not\Leftrightarrow gy \in B_\alpha$, for any $\alpha \in \mathcal{N}$ with $\alpha \sqsupseteq t$, and so $\alpha \in C$ and $\alpha \sqsupseteq s$. Hence C is dense. $\dashv$

By the claim, $\exists \alpha \in \mathcal{N}$ such that $D = \{z \in X : \forall x, y \in [z]_G$ with $x \neq y$, $G <B_\alpha>$ separates x and $y\}$ is comeager and clearly invariant, which completes the proof. □

Theorem 8.2. *Let X be a Polish G-space. If X is aperiodic, then there exists an invariant dense G_δ set that admits a Borel 4-generator.*

Proof. Let A and D be provided by Lemma 8.1. Throwing away an invariant meager set from D, we may assume that D is dense G_δ and hence Polish in the relative topology. Therefore, we may assume without loss of generality that $X = D$.

Thus A aperiodically separates X and hence, by 6.3, there is a partition $\{A_n\}_{n\in\mathbb{N}}$ of X into F_A-invariant Borel complete sections (the latter could be inferred directly from Corollary 6.9

without using Lemma 8.1). Fix an enumeration $G = \{g_n\}_{n\in\mathbb{N}}$ and a countable basis $\{U_n\}_{n\in\mathbb{N}}$ for X. Denote $\mathcal{N}_2 = (\mathbb{N}^2)^{\mathbb{N}}$ and for each $\alpha \in \mathcal{N}_2$, define

$$B_\alpha = \bigcup_{n\geq 1}(A_n \cap g_{(\alpha(n))_0} U_{(\alpha(n))_1}).$$

Claim. $\forall^* \alpha \in \mathcal{N}_2 \forall^* x \in X \forall l \in \mathbb{N} \exists n, k \in \mathbb{N}(\alpha(n) = (k, l) \wedge g_k x \in A_n)$.

Proof of Claim. By Kuratowski-Ulam, it is enough to show that $\forall x \in X$ and $\forall l \in \mathbb{N}$, $C = \{\alpha \in \mathcal{N}_2 : \exists k, n \in \mathbb{N}(\alpha(n) = (k, l) \wedge g_k x \in A_n)\}$ is dense open.

To see that C is open, note that for fixed $n, k, l \in N$, $\alpha(n) = (k, l)$ is an open condition in $\mathcal{N}_2$.

For the density of C, let $s \in (\mathbb{N}^2)^{<\mathbb{N}}$ and set $n = |s|$. Since A_n is a complete section, $\exists k \in \mathbb{N}$ with $g_k x \in A_n$. Any $\alpha \in \mathcal{N}_2$ with $\alpha \sqsupseteq s$ and $\alpha(n) = (k, l)$ belongs to C. Hence C is dense. $\dashv$

By the claim, there exists $\alpha \in \mathcal{N}_2$ such that $Y = \{x \in X : \forall l \in \mathbb{N}\ \exists k, n \in \mathbb{N}\ (\alpha(n) = (k, l) \wedge g_k x \in A_n)\}$ is comeager. Throwing away an invariant meager set from Y, we can assume that Y is G-invariant dense G_δ.

Let $\mathcal{I} = <A, B_\alpha>$, and so $|\mathcal{I}| \leq 4$. We show that $\mathcal{I}$ is a generator on Y. Fix distinct $x, y \in Y$. If x and y are separated by $G <A>$ then we are done, so assume otherwise, that is $x F_A y$. Let $l \in \mathbb{N}$ be such that $x \in U_l$ but $y \notin U_l$. Then there exists $k, n \in \mathbb{N}$ such that $\alpha(n) = (k, l)$ and $g_k x \in A_n$. Since $g_k x F_A g_k y$ and A_n is F_A-invariant, $g_k y \in A_n$. Furthermore, since $g_k x \in A_n \cap g_k U_l$ and $g_k y \notin A_n \cap g_k U_l$, $g_k x \in B_\alpha$ while $g_k y \notin B_\alpha$. Hence $G <B_\alpha>$ separates x and y, and thus so does $G\mathcal{I}$. Therefore $\mathcal{I}$ is a generator. $\square$

Corollary 8.3. *Let X be a Polish G-space. If X is aperiodic, then it is 2-compressible modulo MEAGER.*

Proof. By Theorem 13.1 in [KM04], X is compressible modulo MEAGER. Also, by the above theorem, X admits a 4-generator modulo MEAGER. Thus 2.34 implies that X is 2-compressible modulo MEAGER. $\square$

9. Locally weakly wandering sets and other special cases

Assume throughout the section that X is a Borel G-space.

Definition 9.1. *We say that $A \subseteq X$ is*
- *weakly wandering with respect to $H \subseteq G$ if $(hA) \cap (h'A) = \emptyset$, for all distinct $h, h' \in H$;*
- *weakly wandering, if it is weakly wandering with respect to an infinite subset $H \subseteq G$ (by shifting H, we can always assume $1_G \in H$);*
- *locally weakly wandering if for every $x \in X$, $A^{[x]_G}$ is weakly wandering.*

For $A \subseteq X$ and $x \in A$, put

$$\Delta_A(x) = \{(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}} : g_0 = 1_G \wedge \forall n \neq m(g_n A^{[x]_G} \cap g_m A^{[x]_G} = \emptyset)\},$$

and let $F(G^{\mathbb{N}})$ denote the Effros space of $G^{\mathbb{N}}$, i.e. the standard Borel space of closed subsets of $G^{\mathbb{N}}$ (see 12.C in [Kec95]).

Proposition 9.2. *Let $A \in \mathfrak{B}(X)$.*
(a) $\forall x \in X$, $\Delta_A(x)$ is a closed set in $G^{\mathbb{N}}$.
(b) $\Delta_A : A \to F(G^{\mathbb{N}})$ is $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable and hence universally measurable.

(c) Δ_A *is* F_A*-invariant, i.e.* $\forall x, y \in A$*, if* xF_Ay *then* $\Delta_A(x) = \Delta_A(y)$*.*

(d) If $s : F(G^{\mathbb{N}}) \to G^{\mathbb{N}}$ *is a Borel selector (i.e.* $s(F) \in F$*,* $\forall F \in F(G^{\mathbb{N}})$*), then* $\gamma := s \circ \Delta_A$ *is a* $\sigma(\mathbf{\Sigma}^1_1)$*-measurable* F_A*- and* G*-invariant travel guide. In particular,* A *is a* 1*-traveling set with* $\sigma(\mathbf{\Sigma}^1_1)$*-pieces.*

Proof. (a) $\Delta_A(x)^c$ is open since being in it is witnessed by two coordinates.

(b) For $s \in G^{<\mathbb{N}}$, let $B_s = \{F \in F(G^{\mathbb{N}}) : F \cap V_s \neq \emptyset\}$, where $V_s = \{\alpha \in G^{\mathbb{N}} : \alpha \sqsupseteq s\}$. Since $\{B_s\}_{s \in G^{<\mathbb{N}}}$ generates the Borel structure of $F(G^{\mathbb{N}})$, it is enough to show that $\Delta_A^{-1}(B_s)$ is analytic, for every $s \in G^{<\mathbb{N}}$. But $\Delta_A^{-1}(B_s) = \{x \in X : \exists (g_n)_{n\in\mathbb{N}} \in V_s[g_0 = 1_G \wedge \forall n \neq m g_n(A^{[x]_G} \cap g_m A^{[x]_G} = \emptyset)]\}$ is clearly analytic.

(c) Assume for contradiction that xF_Ay, but $\Delta_A(x) \neq \Delta_A(y)$ for some $x, y \in A$. We may assume that there is $(g_n)_{n\in\mathbb{N}} \in \Delta_A(x) \setminus \Delta_A(y)$ and thus $\exists n \neq m$ such that $g_n A^{[y]_G} \cap g_m A^{[y]_G} \neq \emptyset$. Hence $A^{[y]_G} \cap g_n^{-1} g_m A^{[y]_G} \neq \emptyset$ and let $y', y'' \in A^{[y]_G}$ be such that $y'' = g_n^{-1} g_m y'$. Let $g \in G$ be such that $y' = gy$.

Since $y' = gy$, $y'' = g_n^{-1} g_m gy$ are in A, xF_Ay, and A is F_A-invariant, $gx, g_n^{-1} g_m gx$ are in A as well. Thus $A^{[x]_G} \cap g_n^{-1} g_m A^{[x]_G} \neq \emptyset$, contradicting $g_n A^{[y]_G} \cap g_m A^{[y]_G} = \emptyset$ (this holds since $(g_n)_{n\in\mathbb{N}} \in \Delta_A(x)$).

(d) Follows from parts (b) and (c), and the definition of Δ_A. □

Theorem 9.3. *Let* X *be a Borel* G*-space. If there is a locally weakly wandering Borel complete section for* X*, then* X *admits a Borel* 4*-generator.*

Proof. By part (d) of 9.2 and 2.25, X is 1-compressible. Thus, by 2.30, X admits a Borel 2^2-finite generator. □

Observation 9.4. *Let* $A = \bigcup_{n\in\mathbb{N}} W_n$*, where each* W_n *is weakly wandering and put* $W'_n = W_n \setminus \bigcup_{i<n} [W_i]_G$*. Then* $A' := \bigcup_{n\in\mathbb{N}} W'_n$ *is locally weakly wandering and* $[A]_G = [A']_G$*.*

Corollary 9.5. *Let* X *be a Borel* G*-space. If* X *is the saturation of a countable union of weakly wandering Borel sets,* X *admits a Borel* 3*-generator.*

Proof. Let $A = \bigcup_{n\in\mathbb{N}} W_n$, where each W_n is weakly wandering. By 9.4, we may assume that $[W_n]_G$ are pairwise disjoint and hence A is locally weakly wandering. Using countable choice, take a function $p : \mathbb{N} \to G^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $p(n) \in \bigcap_{x \in W_n} \Delta_{W_n}(x)$ (we know that $\bigcap_{x\in W_n} \Delta_{W_n}(x) \neq \emptyset$ since W_n is weakly wandering).

Define $\gamma : A \to G^{\mathbb{N}}$ by

$$x \mapsto \text{the smallest } k \text{ such that } p(k) \in \Delta_A(x).$$

The condition $p(k) \in \Delta_A(x)$ is Borel because it is equivalent to $\forall n, m \in \mathbb{N}, y, z \in A \cap [x]_G, p(k)(n)y = p(k)(m)z \Rightarrow n = m \wedge x = y$; thus γ is a Borel function. Note that γ is a travel guide for A by definition. Moreover, it is F_A-invariant because if $\Delta_A(x) = \Delta_A(y)$ for some $x, y \in A$, then conditions $p(k) \in \Delta_A(x)$ and $p(k) \in \Delta_A(y)$ hold or fail together. Since Δ_A is F_A-invariant, so is γ. Hence, Lemma 2.29 applied to $\mathcal{I} =<A>$ gives a Borel $(2 \cdot 2 - 1)$-generator. □

Remark. The above corollary in particular implies the existence of a 3-generator in the presence of a weakly wandering Borel complete section. (For a direct proof of this, note that if W is a complete section that is weakly wandering with respect to $\{g_n\}_{n\in\mathbb{N}}$ with $g_0 = 1_G$ and $\{U_n\}_{n\in\mathbb{N}}$ is a family generating the Borel sets, then $\mathcal{I} =< W, \bigcup_{n\geq 1} g_n(W \cap U_n) >$ is a

generator and $|\mathcal{I}| = 3$.) This can be viewed as a Borel version of the Krengel-Kuntz theorem (see 1.9) in the sense that it implies a version of the latter (our result gives a 3-generator instead of a 2-generator). To see this, let X be a Borel G-space and μ be a quasi-invariant measure on X such that there is no invariant measure absolutely continuous with respect to μ. Assume first that the action is ergodic. Then by the Hajian-Kakutani-Itô theorem, there exists a weakly wandering set W with $\mu(W) > 0$. Thus $X' = [W]_G$ is conull and admits a 3-generator by the above, so X admits a 3-generator modulo μ-NULL.

For the general case, one can use Ditzen's Ergodic Decomposition Theorem for quasi-invariant measures (Theorem 5.2 in [Mil08]), apply the previous result to μ-a.e. ergodic piece, combine the generators obtained for each piece into a partition of X (modulo μ-NULL) and finally apply Theorem 6.10 to obtain a finite generator for X. Each of these steps requires a certain amount of work, but we will not go into the details.

Example 9.6. Let $X = \mathcal{N}$ (the Baire space) and $\tilde{E}_0$ be the equivalence relation of eventual agrement of sequences of natural numbers. We find a countable group G of homeomorphisms of X such that $E_G = \tilde{E}_0$. For all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t|$, let $\phi_{s,t} : X \to X$ be defined as follows:

$$\phi_{s,t}(x) = \begin{cases} t \frown y & \text{if } x = s \frown y \\ s \frown y & \text{if } x = t \frown y \\ x & \text{otherwise} \end{cases},$$

and let G be the group generated by $\{\phi_{s,t} : s, t \in \mathbb{N}^{<\mathbb{N}}, |s| = |t|\}$. It is clear that each $\phi_{s,t}$ is a homeomorphism of X and $E_G = \tilde{E}_0$. Now for $n \in \mathbb{N}$, let $X_n = \{x \in X : x(0) = n\}$ and let $g_n = \phi_{0,n}$. Then X_n are pairwise disjoint and $g_n X_0 = X_n$. Hence X_0 is a weakly wandering set and thus X admits a Borel 3-generator by Corollary 9.5.

Example 9.7. Let $X = 2^{\mathbb{N}}$ (the Cantor space) and E_t be the tail equivalence relation on X, that is $xE_t y \Leftrightarrow (\exists n, m \in \mathbb{N})(\forall k \in \mathbb{N}) x(n+k) = y(m+k)$. Let G be the group generated by $\{\phi_{s,t} : s, t \in 2^{<\mathbb{N}}, s \perp t\}$, where $\phi_{s,t}$ are defined as above. To see that $E_G = E_t$ fix $x, y \in X$ with $xE_t y$. Thus there are nonempty $s, t \in 2^{<\mathbb{N}}$ and $z \in X$ such that $x = s \frown z$ and $y = t \frown z$. If $s \perp t$, then $y = \phi_{s,t}(x)$. Otherwise, assume say $s \sqsubseteq t$ and let $s' \in 2^{<\mathbb{N}}$ be such that $s \perp s'$ (exists since $s \neq \emptyset$). Then $s' \perp t$ and $y = \phi_{s',t} \circ \phi_{s,s'}(x)$.

Now for $n \in \mathbb{N}$, let $s_n = \underbrace{11...1}_{n}0$ and $X_n = \{x \in X : x = s_n \frown y, \text{ for some } y \in X\}$. Note that s_n are pairwise incompatible and hence X_n are pairwise disjoint. Letting $g_n = \phi_{s_0,s_n}$, we see that $g_n X_0 = X_n$. Thus X_0 is a weakly wandering set and hence X admits a Borel 3-generator.

Using the function Δ defined above, we give another proof of Proposition 2.27.

Proposition 2.27. *Let X be an aperiodic Borel G-space and $T \subseteq X$ be Borel. If T is a partial transversal then T is $<T>$-traveling.*

Proof. By definition, T is locally weakly wandering.

Claim. *Δ_T is Borel.*

Proof of Claim. Using the notation of the proof of part (b) of 9.2, it is enough to show that $\Delta_T^{-1}(B_s)$ is Borel for every $s \in G^{<\mathbb{N}}$. But since $\forall x \in T$, $T \cap [x]_G$ is a singleton, $\Delta_T(x) \in B_s$ is equivalent to $s(0) = 1_G \wedge (\forall n < m < |s|)\ s(m)x \neq s(n)x$. The latter condition is Borel, hence so is $\Delta_T^{-1}(B_s)$. $\dashv$

By part (d) of 9.2, $\gamma = s \circ \Delta_T$ is a Borel F_T-invariant travel guide for T. $\square$

Corollary 9.8. *Let X be a Borel G-space. If X is smooth and aperiodic, then it admits a Borel 3-generator.*

Proof. Since the G-action is smooth, there exists a Borel transversal $T \subseteq X$. By 2.27, T is $<T>$-traveling. Thus, by 2.29, there is a Borel $(2 \cdot 2 - 1)$-generator. $\square$

Lastly, in case of smooth free actions, a direct construction gives the optimal result as the following proposition shows.

Proposition 9.9. *Let X be a Borel G-space. If the G-action is free and smooth, then X admits a Borel 2-generator.*

Proof. Let $T \subseteq X$ be a Borel transversal. Also let $G \setminus \{1_G\} = \{g_n\}_{n \in \mathbb{N}}$ be such that $g_n \neq g_m$ for $n \neq m$. Because the action is free, $g_n T \cap g_m T = \emptyset$ for $n \neq m$.

Define $\pi : \mathbb{N} \to \mathbb{N}$ recursively as follows:

$$\pi(n) = \begin{cases} \min\{m : g_m \notin \{g_{\pi(i)} : i < n\}\} & \text{if } n = 3k \\ \min\{m : g_m, g_m g_k \notin \{g_{\pi(i)} : i < n\}\} & \text{if } n = 3k+1 \\ \text{the unique } l \text{ s.t. } g_l = g_{\pi(3k+1)} g_k & \text{if } n = 3k+2 \end{cases}.$$

Note that π is a bijection. Fix a countable family $\{U_n\}_{n \in \mathbb{N}}$ generating the Borel sets and put $A = \bigcup_{k \in \mathbb{N}} g_{\pi(3k)}(T \cap U_k) \cup \bigcup_{k \in \mathbb{N}} g_{\pi(3k+1)} T$. Clearly A is Borel and we show that $\mathcal{I} = <A>$ is a generator. Fix distinct $x, y \in X$. Note that since T is a complete section, we can assume that $x \in T$.

First assume $y \in T$. Take k with $x \in U_k$ and $y \notin U_k$. Then $g_{\pi(3k)} x \in g_{\pi(3k)}(T \cap U_k) \subseteq A$ and $g_{\pi(3k)} y \in g_{\pi(3k)}(T \setminus U_k)$. However $g_{\pi(3k)}(T \setminus U_k) \cap A = \emptyset$ and hence $g_{\pi(3k)} y \notin A$.

Now suppose $y \notin T$. Then there exists $y' \in T^{[y]_G}$ and k such that $g_k y' = y$. Now $g_{\pi(3k+1)} x \in g_{\pi(3k+1)} T \subseteq A$ and $g_{\pi(3k+1)} y = g_{\pi(3k+1)} g_k y' = g_{\pi(3k+2)} y' \in g_{\pi(3k+2)} T$. But $g_{\pi(3k+2)} T \cap A = \emptyset$, hence $g_{\pi(3k+1)} y \notin A$. $\square$

Corollary 9.10. *Let H be a Polish group and G be a countable subgroup of H. If G admits an infinite discrete subgroup, then the translation action of G on H admits a 2-generator.*

Proof. Let G' be an infinite discrete subgroup of G. Clearly, it is enough to show that the translation action of G' on H admits a 2-generator. Since G' is discrete, it is closed. Indeed, if d is a left-invariant compatible metric on H, then $B_d(1_H, \epsilon) \cap G' = \{1_H\}$, for some $\epsilon > 0$. Thus every d-Cauchy sequence in G' is eventually constant and hence G' is closed. This implies that the translation action of G' on H is smooth and free (see 12.17 in [Kec95]), and hence 9.9 applies. $\square$

10. A condition for non-existence of non-meager weakly wandering sets

Throughout this section let X be a Polish $\mathbb{Z}$-space and T be the homeomorphism corresponding to the action of $1 \in \mathbb{Z}$.

Observation 10.1. *Let $A \subseteq X$ be weakly wandering with respect to $H \subseteq \mathbb{Z}$. Then A is weakly wandering with respect to*

(a) any subset of H;
(b) $r + H$, $\forall r \in \mathbb{Z}$;
(c) $-H$.

Definition 10.2. *Let $d \geq 1$ and $F = \{n_i\}_{i<k} \subseteq \mathbb{Z}$, where $n_0 < n_1 < \ldots < n_{k-1}$ are increasing. F is called d-syndetic if $n_{i+1} - n_i \leq d$ for all $i < k-1$. In this case we say that the length of F is $n_{k-1} - n_0$ and denote it by $||F||$.*

Lemma 10.3. *Let $d \geq 1$ and $F \subseteq \mathbb{Z}$ be a d-syndetic set. For any $H \subseteq \mathbb{Z}$, if $|H| = d+1$ and $\max(H) - \min(H) < ||F|| + d$, then F is not weakly wandering with respect to H (viewing $\mathbb{Z}$ as a $\mathbb{Z}$-space).*

Proof. Using (b) and (c) of 10.1, we may assume that H is a set of non-negative numbers containing 0. Let $F = \{n_i\}_{i<k}$ with n_i increasing.

Claim. *$\forall h \in H$, $(h + F) \cap [n_{k-1}, n_{k-1} + d) \neq \emptyset$.*

Proof of Claim. Fix $h \in H$. Since $0 \leq h < ||F|| + d$,

$$n_0 + h < n_0 + (||F|| + d) = n_{k-1} + d.$$

We prove that there is $0 \leq i \leq k-1$ such that $n_i + h \in [n_{k-1}, n_{k-1} + d)$. Otherwise, because $n_{i+1} - n_i \leq d$, one can show by induction on i that $n_i + h < n_{k-1}, \forall i < k$, contradicting $n_{k-1} + h \geq n_{k-1}$. ⊣

Now $|H| = d + 1 > d = |\mathbb{Z} \cap [n_{k-1}, n_{k-1} + d)|$, so by the Pigeon Hole Principle there exists $h \neq h' \in H$ such that $(h + F) \cap (h' + F) \neq \emptyset$ and hence F is not weakly wandering with respect to H. □

Definition 10.4. *Let $d, l \geq 1$ and $A \subseteq X$. We say that A contains a d-syndetic set of length l if there exists $x \in X$ such that $\{n \in \mathbb{Z} : T^n(x) \in A\}$ contains a d-syndetic set of length $\geq l$. This is equivalent to $\bigcap_{n \in F} T^n(A) \neq \emptyset$, for some d-syndetic set $F \subseteq \mathbb{Z}$ of length $\geq l$.*

For $A \subseteq X$, define $s_A : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by

$$d \mapsto \sup\{l \in \mathbb{N} : A \text{ contains a } d\text{-syndetic set of length } l\}.$$

Also, for infinite $H \subseteq \mathbb{Z}$, define a width function $w_H : \mathbb{N} \to \mathbb{N}$ by

$$d \mapsto \min\{\max(H') - \min(H') : H' \subseteq H \wedge |H'| = d + 1\}.$$

Proposition 10.5. *If $A \subseteq X$ is weakly wandering with respect to an infinite $H \subseteq \mathbb{Z}$ then $\forall d \in \mathbb{N}, s_A(d) + d \leq w_H(d)$.*

Proof. Let H be an infinite subset of $\mathbb{Z}$ and $A \subseteq X$, and assume that $s_A(d) + d > w_H(d)$ for some $d \in \mathbb{N}$. Thus $\exists x \in X$ such that $\{n \in \mathbb{Z} : T^n(x) \in A\}$ contains a d-syndetic set F of length l with $l + d > w_H(d)$ and $\exists H' \subseteq H$ such that $|H'| = d + 1$ and $\max(H') - \min(H') = w_H(d)$. By Lemma 10.3 applied to F and H', F is not weakly wandering with respect to H' and hence neither is A. Thus A is not weakly wandering with respect to H. □

Corollary 10.6. *If $A \subseteq X$ contains arbitrarily long d-syndetic sets for some $d \geq 1$, then it is not weakly wandering.*

Proof. If A and d are as in the hypothesis, then $s_A(d) = \infty$ and hence, by Proposition 10.5, A is not weakly wandering with respect to any infinite $H \subseteq \mathbb{Z}$. □

Theorem 10.7. *Let X be a Polish G-space. Suppose for every nonempty open $V \subseteq X$ there exists $d \geq 1$ such that V contains arbitrarily long d-syndetic sets, i.e. $\bigcap_{n \in F} T^n(V) \neq \emptyset$ for arbitrarily long d-syndetic sets $F \subseteq \mathbb{Z}$. Then X does not admit a non-meager Baire measurable weakly wandering subset.*

Proof. Let A be a non-meager Baire measurable subset of X. By the Baire property, there exists a nonempty open $V \subseteq X$ such that A is comeager in V. By the hypothesis, there exists arbitrarily long d-syndetic sets $F \subseteq \mathbb{Z}$ such that $\bigcap_{n \in F} T^n(V) \neq \emptyset$. Since A is comeager in V and T is a homeomorphism, $\bigcap_{n \in F} T^n(A)$ is comeager in $\bigcap_{n \in F} T^n(V)$, and hence $\bigcap_{n \in F} T^n(A) \neq \emptyset$ for any F for which $\bigcap_{n \in F} T^n(V) \neq \emptyset$. Thus A also contains arbitrarily long d-syndetic sets and hence, by Corollary 10.6, A is not weakly wandering. □

Corollary 10.8. *Let X be a Polish G-space. Suppose for every nonempty open $V \subseteq X$ there exists $d \geq 1$ such that $\{T^{nd}(V)\}_{n \in \mathbb{N}}$ has the finite intersection property. Then X does not admit a non-meager Baire measurable weakly wandering subset.*

Proof. Fix nonempty open $V \subseteq X$ and let $d \geq 1$ such that $\{T^{nd}(V)\}_{n \in \mathbb{N}}$ has the finite intersection property. Then for every N, $F = \{kd : k \leq N\}$ is a d-syndetic set of length Nd and $\bigcap_{n \in F} T^n(V) \neq \emptyset$. Thus Theorem 10.7 applies. □

Lemma 10.9. *Let X be a generically ergodic Polish G-space. If there is a non-meager Baire measurable locally weakly wandering subset then there is a non-meager Baire measurable weakly wandering subset.*

Proof. Let A be a non-meager Baire measurable locally weakly wandering subset. By generic ergodicity, we may assume that $X = [A]_G$. Throwing away a meager set from A we can assume that A is G_δ. Then, by (d) of 9.2, there exists a $\sigma(\mathbf{\Sigma}_1^1)$-measurable (and hence Baire measurable) G-invariant travel guide $\gamma : A \to G^{\mathbb{N}}$. By generic ergodicity, γ must be constant on a comeager set, i.e. there is $(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ such that $Y := \gamma^{-1}((g_n)_{n \in \mathbb{N}})$ is comeager. But then $W := A \cap Y$ is non-meager and is weakly wandering with respect to $\{g_n\}_{n \in \mathbb{N}}$. □

Let $X = \{\alpha \in 2^{\mathbb{N}} : \alpha \text{ has infinitely many 0-s and 1-s}\}$ and T be the odometer transformation on X. We will refer to this $\mathbb{Z}$-space as the odometer space.

Corollary 10.10. *The odometer space does not admit a non-meager Baire measurable locally weakly wandering subset.*

Proof. Let $\{U_s\}_{s \in 2^{<\mathbb{N}}}$ be the standard basis. Then for any $s \in 2^{<\mathbb{N}}$, $T^d(U_s) = U_s$ for $d = |s|$. Thus $\{T^{nd}(U_s)\}_{n \in \mathbb{N}}$ has the finite intersection property, in fact $\bigcap_{n \in \mathbb{N}} T^{nd}(U_s) = U_s$. Hence, we are done by 10.8 and 10.9. □

The following corollary shows the failure of the analogue of the Hajian-Kakutani-Itô theorem in the context of Baire category as well as gives a negative answer to Question 1.10.

Corollary 10.11. *There exists a generically ergodic Polish $\mathbb{Z}$-space Y (namely an invariant dense G_δ subset of the odometer space) with the following properties:*

(i) there does not exist an invariant Borel probability measure on Y;
(ii) there does not exist a non-meager Baire measurable locally weakly wandering set;
(iii) there does not exist a Baire measurable countably generated partition of Y into invariant sets, each of which admits a Baire measurable weakly wandering complete section.

Proof. By the Kechris-Miller theorem (see 1.12), there exists an invariant dense G_δ subset Y of the odometer space that does not admit an invariant Borel probability measure. Now (ii) is asserted by Corollary 10.10. By generic ergodicity of Y, for any Baire measurable countably generated partition of Y into invariant sets, one of the pieces of the partition has to be comeager. But then that piece does not admit a Baire measurable weakly wandering complete section since otherwise it would be non-meager, contradicting (ii). □

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